- Finish the HMW2

Definition. X, Y normed spaces and $T : X \to Y$ isometry. Assume that T is onto then T is called isometric isomorphism and X and Y are called isometrically isomorphic.

Theorem (Riesz-Fischer)
H Hilbert space infinite dimensional and separable. Consider $\{e_n\} \subset H$ an o.n.b. for H. The the operator $T : H \to \ell^2$ defined by

$$Tx = ((x, e_n))_n, \quad \forall x \in H \qquad (RF)$$

is an isometric isomorphism.

Proof. Since $x \in H$, by Parseval's Theorem

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 < +\infty$$

and $\|x\| = \|((x, e_n))_n\|_{\ell^2}$

So $T : H \to \ell^2$ well defined and T is an isometry.

T is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall x_1, x_2 \in H$,

$$T(\alpha x_1 + \beta x_2) = ((\alpha x_1 + \beta x_2, e_n))_n = (\alpha(x_1, e_n) + \beta(x_2, e_n))_n$$
$$= \alpha((x_1, e_n))_n + \beta((x_2, e_n))_n$$
$$= \alpha T x_1 + \beta T x_2$$

T is onto: $\forall (\beta_n)_n \in \ell^2$ the series $\sum_n \beta_n e_n$ is convergent to $y \in H$; then $Ty = (\beta_n)_n$. In fact,

$$Ty = \Big(\big(\underbrace{\sum_{n=1}^{\infty} \beta_n e_n}_{=y}, e_m\big)\Big)_m$$

$$\underbrace{\Big(\sum_{n=1}^{\infty} \beta_n e_n, e_m\Big)}_{=(y, e_m)} \overset{\text{cont.}(\cdot,\cdot)}{=} \sum_{n=1}^{\infty} \beta_n \underbrace{(e_n, e_m)}_{\delta_{nm}} = \beta_m$$

$\forall m \in \mathbb{N}_+$, hence $\beta_n = (y, e_n)$, $\forall n$.

Hence T is a bijection and so an isometric isomorphism. ■

$\forall m \in \mathbb{N}_+$, $\quad Te_m = \big((e_m, e_n)\big)_{n \in \mathbb{N}_+} = (\delta_{mn})_n$
$= \delta_m$

$\{\delta_m\}_{m \in \mathbb{N}_+}$ is an o.n.b. for ℓ^2

Hence T maps the o.n.b. $\{e_n\}$ of H onto the o.n.b. $\{\delta_n\} \subset \ell^2$.

Corollary. Any infinite-dimensional and separable Hilbert space is isometrically isomorphic to ℓ^2.

Property. X, Y inner product spaces, $T: X \to Y$ isometry. Then $(Tx, Ty)_Y = (x, y)_X$, $\quad \forall x, y \in X$.

Proof. If $X = \mathbb{R}$ then the polarization identity

$$(x, y) = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}$$

if $X = \mathbb{C}$, $\quad (x, y) = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$

So, if $X = \mathbb{R}$ $\quad (Tx, Ty)_Y \overset{\text{p.i.}}{=} \frac{1}{4}\{\|Tx+Ty\|_Y^2 - \|Tx-Ty\|_Y^2\}$

$$\overset{\text{lin}}{=} \frac{1}{4}\{\|T(x+y)\|_Y^2 - \|T(x-y)\|_Y^2\}$$

T isometry $\overset{=}{} \frac{1}{4}\{\|x+y\|_X^2 - \|x-y\|_X^2\}$
$\overset{p.i.}{=} (x,y)_X$

Similarly for $X = \mathbb{C}$.

Theorem. X normed space, Y Banach space then $B(X,Y)$ is a Banach space.

Proof. Consider any Cauchy seq. $\{T_n\} \subset B(X,Y)$, we want to prove that $\{T_n\}$ is convergent in $B(X,Y)$. $\forall x \in X$,

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_{B(X,Y)} \|x\|_X$$

$\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} : \forall n, m \geq N_0 \ \|T_n - T_m\| < \varepsilon$

$\{T_n x\} \subset Y$ is a Cauchy seq. in Y, since Y is complete $\{T_n x\}$ is convergent and we define $Tx := \lim_{n\to\infty} T_n x \in Y$

By construction, $T : X \to Y$

- T is linear, $\forall \alpha, \beta \in \mathbb{F}$, $\forall x_1, x_2 \in X$

$$T(\alpha x_1 + \beta x_2) = \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2)$$
$$= \lim_{n\to\infty} [\alpha T_n x_1 + \beta T_n x_2]$$
$$= \alpha \lim_{n\to\infty} T_n x_1 + \beta \lim_{n\to\infty} T_n x_2$$
$$= \alpha T x_1 + \beta T x_2$$

$T_n \to T$ in $B(X,Y)$; we know:

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\|_{B(X,Y)} \|x\|_X, \quad \forall x \in X$$

$\forall \varepsilon > 0 \quad \exists N_0 \mid \quad \forall n, m \geq N_0 \qquad \|T_n - T_m\|_{B(X,Y)} < \varepsilon$

hence

$$\|T_n x - T_m x\|_Y < \varepsilon \|x\|_X , \qquad \forall x$$

take the limit for $m \to +\infty$

$$\|T_n x - \underbrace{\lim_{m\to\infty} T_m x}_{Tx}\|_Y \leq \varepsilon \|x\|_X , \qquad \forall x$$

$$\|T_n x - T x\|_Y \leq \varepsilon \|x\|_X , \qquad \forall x$$

$$\|(T_n - T) x\|_Y \leq \varepsilon \|x\|_X , \quad \forall x , \qquad \forall n \geq N_0$$

$$\|T_n - T\|_{B(X,Y)} \leq \varepsilon , \qquad \forall n \geq N_0$$

$$T_n \to T \qquad \text{in } B(X,Y) .$$

$$\|T\|_{B(X,Y)} = \|(T - T_n) + T_n\|_{B(X,Y)}$$
$$\leq \|T - T_n\|_{B(X,Y)} + \underbrace{\|T_n\|_{B(X,Y)}}_{<\infty}$$

fix an $n \geq N_0$, hence $\|T\|_{B(X,Y)} < \infty$, that is T is bounded. ▣

Definition. X normed space. The **dual space** of X is $X' := B(X, \mathbb{F})$.

Any linear operator $T : X \to \mathbb{F}$ is called **functional**.

Another notation for X' is X^*.

Corollary. If X is a normed space then X' is a Banach space.

Proof. $\mathbb{F}$ is complete so the result follows from the previous Theorem.

Lemma. $T \in B(X,Y)$, $S \in B(Y,Z)$, with X, Y, Z normed spaces. Then $S \circ T \in B(X,Z)$ and

$$\|S \circ T\|_{B(X,Z)} \leq \|S\|_{B(Y,Z)} \|T\|_{B(X,Y)} .$$

Proof. $S \circ T(\alpha x_1 + \beta x_2) \overset{T \text{ lin}}{=} S[\alpha T x_1 + \beta T x_2] \overset{S \text{ lin}}{=} \alpha (S \circ T) x_1 + \beta (S \circ T) x_2$

$\forall x \in X$ $\quad \|(S \circ T) x\|_Z = \|S[Tx]\|_Z \overset{S \text{ bound.}}{\leq} \|S\| \|Tx\|_Y \overset{T \text{ bound.}}{\leq} \|S\| \|T\| \|x\|_X$

hence $S \circ T \in B(X,Z)$ and $\|S \circ T\|_{B(X,Z)} \leq \|S\| \|T\|$

Notation. $ST := S \circ T$, called the **product** of the operators S and T.

Lemma. X normed space, $B(X)$ is an algebra with identity (with respect to the product of operators).

Proof. $\forall S, T \in B(X)$ $\quad ST \in B(X)$ by the previous lemma so, since $B(X)$ is a vector space we showed that $B(X)$ is an algebra.

Consider the identity mapping $Ix = x$, $\forall x \in X$

$I \in B(X)$ and $IT = TI = T$, $\forall T \in B(X)$

so I is an identity in $B(X)$.

Notation. $T \in B(X)$, $T^2 := TT$, $T^3 := TTT$, ...,

$$T^n := \underbrace{TT \cdots T}_{n \text{ times}} \in B(X), \quad \forall n \in \mathbb{N}_+$$

If $a_j \in \mathbb{F}$, $0 \le j \le m$, $p: \mathbb{F} \to \mathbb{F}$ a polynomial with coefficients a_j: $p(x) = a_0 + a_1 x + \dots + a_m x^m$

we define by $p(T) = \underbrace{a_0 I + a_1 T + \dots + a_m \underbrace{T^m}_{\in B(X)}}_{\in B(X) \text{ (lin. comb. of bounded op.)}} \in B(X)$

for any $T \in B(X)$

Lemma. $T \in B(X)$, p, q polynomials, $\lambda, \mu \in \mathbb{F}$, then

1) $(\lambda p + \mu q)(T) = \lambda p(T) + \mu q(T)$

sum of polynom. sum of operators

2) $(pq)(T) = p(T) q(T)$

product of polynom. product of operators.

INVERSE OPERATORS

From linear algebra, A $n \times n$ matrix, study

$$Ax = y \qquad (*)$$

- Step 1: check whether A is invertible $\iff \det A \neq 0$
- Step 2: if $\det A \neq 0$ you can solve $(*)$ $x = A^{-1} y$.

Definition. X, Y normed spaces, $T \in B(X, Y)$ is said to be invertible if there exists a $S \in B(Y, X)$ such that $ST = I_X$, $TS = I_Y$. S is called *the inverse of T* and denoted by T^{-1}.

Equivalently, T is invertible if T is a bijection and T^{-1} is bounded.

Example. $I \in B(X)$ is invertible $I^{-1} = I$.

Remark: $(T^{-1})^{-1} = T$ by the definition with $S = T^{-1}$ and changing roles between T and T^{-1}.

Lemma. If $T_1 \in B(X,Y)$, $T_2 \in B(Y,Z)$ are invertible then $T_2 T_1 \in B(X,Z)$ is invertible and the inverse $(T_2 T_1)^{-1} = T_1^{-1} T_2^{-1}$.

Proof. Check as exercise, using the definition.

If $T \in B(X)$ is invertible then by the previous lemma also T^n is invertible, $\forall n \in \mathbb{N}_+$, and

$$(T^n)^{-1} = \underbrace{T^{-1} \cdot T^{-1} \cdots T^{-1}}_{n\text{-times}} = (T^{-1})^n$$

Notation $T^{-n} := (T^n)^{-1}$

Definition. If X, Y normed spaces such that $\exists\, T \in B(X,Y)$ invertible then X and Y are called *isomorphic* whereas T is called *isomorphism* between X and Y.

Theorem. If X, Y are isomorphic, then

1) $\dim X < \infty \iff \dim Y < \infty$ in which case $\dim X = \dim Y$

2) X separable $\iff$ Y separable

3) X complete $\iff$ Y complete.