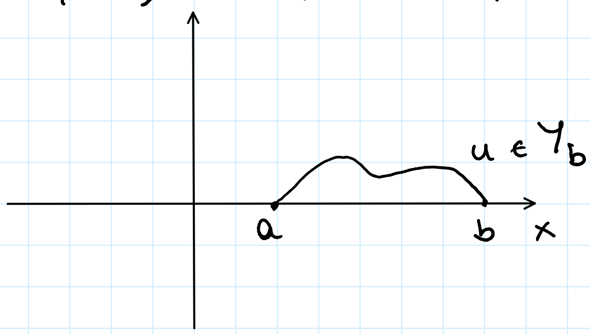
DIFFERENTIAL EQUATIONS

$X = \mathcal{C}([a,b])$, $\|\cdot\|_\infty$.

$H = L^2([a,b])$

$\mathcal{C}^k([a,b]) = \{ f \in \mathcal{C}([a,b]) : f^{(m)} \in \mathcal{C}([a,b]), \ \forall m = 1, \dots, k \}$

$Y_b = \{ u \in \mathcal{C}^2([a,b]) : u(a) = u(b) = 0 \}$



Consider the following boundary value problem (BVP)

$(BVP)_1$ $\begin{cases} u'' = w \\ u(a) = u(b) = 0 \end{cases}$ "boundary conditions"

with $w \in X$.

A solution of the BVP is a function $u \in \mathcal{C}^2([a,b])$ such that $u(a) = u(b) = 0$, that is $u \in Y_b$

Remark. Consider the differential operator $T_b : Y_b \to X$, $u \mapsto u''$, then T_b is not bounded w.r.t. $\|\cdot\|_\infty$

Example. $[a,b] = [0,1]$, $p_m(x) = x^m(1-x)$, $m \in \mathbb{N}_+$

$\{p_m\} \subset Y_b$, $p_m \in \mathcal{C}^\infty([0,1])$, $p_m(0) = 0$, $p_m(1) = 0$

$$\|p_m\|_\infty = \sup_{x \in [0,1]} |p_m(x)| \le 1, \quad \forall m \in \mathbb{N}_+$$

$$p_m'(x) = m x^{m-1}(1-x) - x^m$$

$$p_m''(x) = m(m-1)x^{m-2}(1-x) - m x^{m-1} - m x^{m-1}$$
$$= m\,[\underbrace{(m-1)x^{m-2}(1-x) - 2x^{m-1}}_{=\tilde{p}_m}]$$
$$\|p_m''\|_\infty = m\,\|\tilde{p}_m\|_\infty \longrightarrow \infty, \quad \text{as } m \to \infty$$
hence
$$\underbrace{\|T_b p_m\|_\infty}_{\substack{\downarrow\ m\to\infty \\ \infty}} \le \|T_b\|\,\underbrace{\|p_m\|_\infty}_{\le 1}$$

the estimate does not hold!

T_b is unbounded!.

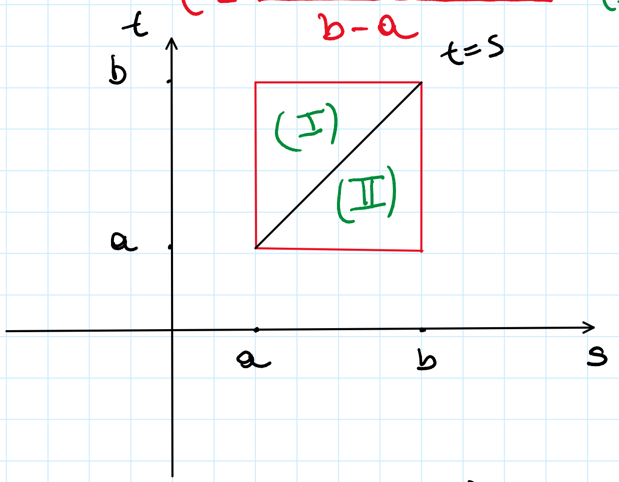
Other approach to the study of $(BVP)_1$: rewrite $(BVP)_1$ in terms of an integral equation!

Exercise. Show that $(BVP)_1$ is uniquely solvable and the solution is given by

$$u = G_0 w, \qquad (G)$$

with G_0 the integral operator with kernel

$$g_0(s,t) = \begin{cases} -\dfrac{(s-a)(b-t)}{b-a} & (\mathrm{I}) \quad a \le s \le t \le b \\[2ex] -\dfrac{(t-a)(b-s)}{b-a} & (\mathrm{II}) \quad a \le t \le s \le b. \end{cases}$$



$g_0(s,t) \in \mathcal{C}([a,b]\times[a,b])$

Hence $G_0 = T_b^{-1}$ only as linear transformation since T_b is not bounded.

Solution of (G): $u(s) = \int_a^b g_0(s,t)\,w(t)\,dt$

observe that $u(a) = 0 = u(b)$

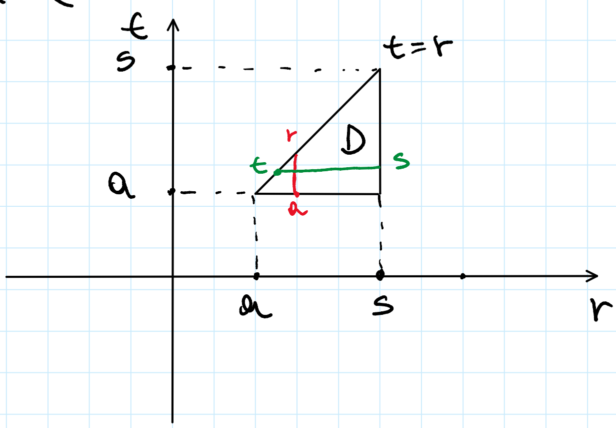
We have check that $u'' = w$

$w \in \mathcal{C}([a,b])$

$$u'(s) = \int_a^s w(t)\,dt + \alpha_1, \qquad \alpha_1 \in \mathbb{C}$$

$$u(s) = \int_a^s \left(\int_a^r w(t)\,dt\right) dr + \int_a^s \alpha_1\,dr + \alpha_2, \quad \alpha_2 \in \mathbb{C}$$

$$= \int_a^s \left(\int_a^r w(t)\,dt\right) dr + \alpha_1 (s-a) + \alpha_2$$



D : domain of integration

$$= \int_a^s \left(\int_t^s w(t)\,dr\right) dt + \alpha_1 (s-a) + \alpha_2$$

$$= \int_a^s w(t)\left(\int_t^s dr\right) dt + \alpha_1 (s-a) + \alpha_2$$

$$= \int_a^s (s-t)\,w(t)\,dt + \alpha_1 (s-a) + \alpha_2$$

If we assume $u(a) = 0$

$$\Rightarrow \quad 0 = u(a) = \underbrace{\int_a^a (s-t)\,w(t)\,dt}_{=0} + \alpha_1 \underbrace{(a-a)}_{=0} + \alpha_2$$

$$\Rightarrow \quad \alpha_2 = 0$$

If we assume $u(b) = 0$

$$0 = u(b) = \int_a^b (b-t)\,w(t)\,dt + \alpha_1 (b-a)$$

hence $\qquad \alpha_1 = -\int_a^b \frac{b-t}{b-a}\, w(t)\,dt$

So we have found the unique solution:

$$u(s) = \int_a^s (s-t)\,w(t)\,dt - \int_a^b \frac{(b-t)(s-a)}{b-a}\, w(t)\,dt$$

$$= \int_a^s \left[(s-t) - \frac{(b-t)(s-a)}{b-a} \right] w(t)\,dt \; +$$

$$- \int_s^b \frac{(b-t)(s-a)}{b-a}\, w(t)\,dt$$

$$= \int_a^b g_0(s,t)\, w(t)\,dt$$

Remark. In a sense "T_b is invertible" and its "inverse" is G_0, but only as <u>linear operators</u> since T_b is unbounded.

Definition G_0 is called Green's operator, whereas the kernel g_0 is called Green's function.

Exercise. "Sturm-Liouville problem" (SL)

$q, f \in X$. Consider

$$(SL)_1 \quad \begin{cases} u''(s) + q(s)\,u(s) = f(s) \\ u(a) = u(b) = 0 \qquad \text{boundary conditions} \end{cases}$$

Show that $(SL)_1$ has a solution $u \in Y_b \Leftrightarrow$ u satisfies:

$(*)\quad u(s) + \underbrace{\int_a^b g_0(s,t)q(t)u(t)\,dt}_{= G_0(qu)} = g(s)$, with

$g(s) = G_0 f(s)$.

Solution. We write $(SL)_1$ as follows:

$$\begin{cases} u''(s) = \underbrace{-q(s)u(s) + f(s)}_{= w(s)} \\ u(a) = u(b) = 0 \end{cases} \qquad w \in X$$

By the previous example, the solution is given by

$$u = G_0 w = G_0(-qu + f) \overset{G_0 \text{ lin}}{=} -G_0(qu) + G_0 f$$

hence we get $(*)$:

$$u(s) + G_0(qu)(s) = g(s)$$

If we call $\tilde{g}_0(s,t) = g_0(s,t)q(t)$

then $(*)$ can be rewritten as

$$I u + \tilde{G}_0 u = g \iff (I + \tilde{G}_0)u = g$$

where $\tilde{G}_0$ is the integral operator with kernel $\tilde{g}_0$.

Exercise. Show that

$$(I + \tilde{G}_0)u = g$$

is well-posed if we assume $(b-a)^2 \|q\|_\infty < 4$.

Solution. We will show that the operator $I + \tilde{G}_0$ is invertible. We want to apply the Neumann series theorem and show that $\|\tilde{G}_0\|_{B(X)} < 1$

$$\|\tilde{G}_0 u\|_\infty = \sup_{s \in [a,b]} \left| \int_a^b g_0(s,t)q(t)u(t)\,dt \right|$$

$$\leq \sup_{s\in[a,b]} \int_a^b \underbrace{|g_0(s,t)|}_{\leq \|g_0\|_{L^\infty([a,b]\times[a,b])}} \cdot \underbrace{|q(t)|}_{\leq \|q\|_\infty} \cdot \underbrace{|u(t)|}_{\leq \|u\|_\infty} dt$$

$$\leq \underbrace{(b-a)\,\|g_0\|_{L^\infty([a,b]\times[a,b])}\,\|q\|_\infty}_{\text{we want to prove } <1} \|u\|_\infty$$

By assumption, $\|q\|_\infty < \dfrac{4}{(b-a)^2}$

exercise : $\|g_0\|_{L^\infty([a,b]\times[a,b])} = \sup_{(s,t)\in[a,b]\times[a,b]} |g_0(s,t)|$

$$= \max_{(s,t)\in[a,b]\times[a,b]} |g_0(s,t)| = \frac{b-a}{4}$$

maximum point $(s,t) = \left(\dfrac{a+b}{2}, \dfrac{a+b}{2}\right)$

hence $(b-a)\,\|g_0\|_\infty \|q\|_\infty < (b-a)\dfrac{(b-a)}{4} \cdot \dfrac{4}{(b-a)^2} = 1$

hence $\|\tilde{G}_0\|_{B(X)} < 1$ so that the operator $I + \tilde{G}_0$ is invertible and the solution is given by $u = (I + \tilde{G}_0)^{-1} g$

so the problem is well-posed since $(I + \tilde{G}_0)^{-1} \in B(X)$.

We can consider the "eigenvalue problem" (EP) associated with $(BVP)_1$:

$$\begin{cases} u'' = \lambda u \\ u(a) = u(b) = 0 \end{cases} \qquad (EP)_1$$

An eigenvalue λ is a number $\lambda \in \mathbb{C}$ such that $\exists\, u \in Y_b$, $u \neq 0$, solution to $(EP)_1$.

Any solution $u \in Y_b$, $u \neq 0$, is called an eigenfunction (more common than eigenvector).

Putting $w := \lambda u$ we see that $(EP)_1$ has solution $u = G_0 w = G_0(\lambda u) = \lambda G_0 u$

Thus λ is an eigenvalue of $(EP)_1 \iff \lambda^{-1}$ is an eigenvalue of G_0.

Observe that $\lambda = 0$ cannot be an eigenvalue of $(EP)_1$ since G_0 is linear: $u = G_0 0 = 0$

$\Rightarrow$ the eigenfunctions of $(EP)_1$ coincide with those of G_0.

Remark. If $g_0 \in \mathcal{C}([a,b]\times[a,b]) \subset L^2([a,b]\times[a,b])$

So if we consider $G_0 : H \to H$,

$H = L^2([a,b])$

g_0 is real-valued and is symmetric:

$g_0(s,t) = g_0(t,s) \Rightarrow G_0^* = G_0$ and

it is compact. From the spectral theory for compact self-adjoint operators: $\exists$ a sequence (λ_n) of real eigenvalues and an o.n. sequence of eigenfunctions $\{e_n\}$.

Property. $\operatorname{Ker} G_0 = \{0\}$.

Hence $\{e_n\}$ is an o.n.b. for H.

More generally, the E.P. associated with the $(SL)_1$

$$\begin{cases} u'' + qu = \lambda u \\ u(a) = u(b) = 0 \end{cases} \qquad (EP)_2$$

The spectral properties of $(EP)_2$ are obtained from those of $\tilde{G}_0$.