Analysis (SDS – UNITO, 23/24)
Weekly mix #4

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Exercise 1

Let $1 \le p < \infty$ and $n \in \mathbb{N}$. Consider the family of operators

$$T_n \colon \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}), \qquad (T_n(x))_k := \begin{cases} x_k & (k \le n) \\ x_k + \frac{1}{n} x_{k-n} & (k \ge n+1). \end{cases}$$

(a) Prove that T_n is a linear and bounded operator – that is, $T_n \in \mathcal{B}(\ell^p(\mathbb{N}))$.
In particular, show that

$$\|T_n\|_{\mathcal{B}(\ell^p)} \le 1 + \frac{1}{n}.$$

(b) Prove that $T_n \to I$ in $\mathcal{B}(\ell^p(\mathbb{N}))$, that is

$$\lim_{n\to\infty} \|T_n - I\|_{\mathcal{B}\ell^p} = 0.$$

(c) Prove that

$$\|T_n\|_{\mathcal{B}(\ell^p)} = 1 + \frac{1}{n}.$$

[**Hint.** If $p = 1$ compute $T_n(e_n)$. If $p > 1$ compute $T_n(\sum_{j=1}^k e_j)$ for $k \ge n+1$.]

Exercise 2

Let X be a Banach space and $h \colon [0,1] \to X$ a function such that $f \circ h \in C[0,1]$ for all $f \in X'$.

(a) Prove that h is bounded, that is $\sup_{t\in[0,1]} \|h(t)\|_X < \infty$.

(b) Prove that the functional $\psi \colon X' \to \mathbb{R}$ given by

$$\psi(f) := \int_0^1 f(h(t))dt$$

is linear and continuous.

(c) If X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, show that there exists a unique $v \in X$ such that

$$\langle v, x \rangle = \int_0^1 \langle x, h(t) \rangle dt, \qquad x \in X.$$

Exercise 3

Let H be a Hilbert space, $(f_n)_{n\in\mathbb{N}}$ be a sequence in H and assume that $f \in H$.

Prove that $f_n \to f$ in H if and only if $f_n \rightharpoonup f$ and $\|f_n\|_H \to \|f\|_H$.

Exercise 4

Consider the set

$$X = \{f \in C[0,1] : f(0) = 0\}$$

and the functional

$$T\colon X \to \mathbb{R}, \qquad T(f) := \int_0^1 f(t)dt.$$

(a) Prove that $T \in X'$ and compute $\|T\|_{X'}$.

(b) Discuss whether there exists $f \in X$ such that $\|f\|_\infty = 1$ and $T(f) = \|T\|_{X'}$.

Exercise 5

Consider the function

$$\psi\colon (0,1) \to \mathbb{R}, \qquad \psi(t) := t\mathbf{1}_{(0,1/2]} = \begin{cases} t & (0 < t \leq 1/2) \\ 0 & (1/2 < t < 1). \end{cases}$$

Let then T be the operator on $L^2(0,1)$ defined by $Tf := \psi f$.

Prove that $T \in \mathcal{B}(L^2(0,1))$ and compute $\|T\|_{\mathcal{B}(L^2)}$.

Exercise 6

Consider the set

$$C = \{f \in L^2[0,1] : f \geq 0 \text{ a.e. in } [0,1]\}.$$

(a) Prove that C is a closed convex subset of $L^2[0,1]$.

(b) Discuss whether C is a linear subspace of $L^2[0,1]$.

(c) Determine the explicit form of the orthogonal projection $P_C\colon L^2[0,1] \to C$.

(d) Compute the distance between $f(t) = 2t - 1$ and C.