Elements of Mathematical Logic

Section 16 of Chapter V

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Two fundamental principles.

Extensionality principle

Two sets are equal if and only if they have the same elements.

Thus a set is completely determined by its elements.

Comprehension principle

Any property $P(x)$ defines a set, there is a set $\{x \mid P(x)\}$.

Russell's paradox

Consider property $P(x)$ given by $x \notin x$ and let $R = \{x \mid x \notin x\}$.

$$R \notin R \Rightarrow R \in R \text{ and } R \in R \Rightarrow R \notin R$$

Fix a first order language $\mathcal{L}_\in$ with only a binary predicate $\in$.
The objects of our studies are called **classes**.

Definition

A class A is a **set** iff A belongs to some class. Let $\mathrm{Set}(x)$ be the formula asserting that x is a set: $\exists y(x \in y)$. A **proper class** is a class that is not a set.

We now define the **Morse-Kelly** MK first order theory in the language $\mathcal{L}_\in$.

Extensionality Axiom

$\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y$.

Comprehension Axiom-scheme

If x occurs free in $\varphi(x, y_1, \ldots, y_n)$ and A is a variable distinct from $x, y_1, \ldots, y_n$, then $\exists A \forall x\, (x \in A \Leftrightarrow \exists z(x \in z) \wedge \varphi(x, y_1, \ldots, y_n))$.

In other words: for each φ and every choice of classes $y_1, \ldots, y_n$, there is a class A of all elements satisfying φ. By extensionality A is unique, and it is denoted by

$$A = \{x \mid \varphi(x, y_1, \ldots, y_n)\}$$

Warning:

There are infinitely many axioms, one for each φ.

Back to Russell's paradox: by comprehension $R = \{x \mid x \notin x\}$ is a class. If R were a set, then $R \in R \Leftrightarrow R \notin R$, a contradiction. Therefore R is a proper class.

Notation

- $\{x \in A \mid \varphi(x, y_1, \ldots, y_n)\}$ *is the class given by the formula* $x \in A \wedge \varphi(x, y_1, \ldots, y_n)$.
- $A \cap B = \{x \mid x \in A \wedge x \in B\} = \{x \in A \mid x \in B\} = \{x \in B \mid x \in A\}$.
- $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- $A \setminus B = \{x \mid x \in A \wedge x \notin B\} = \{x \in A \mid x \notin B\}$.
- $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$.

By the Extensionality Axiom $A \cap B = B \cap A$, $A \cup B = B \cup A$ and $A \bigtriangleup B = B \bigtriangleup A$.

Set Existence Axiom

$\exists x \exists y\, (x \in y)$.

Definition

x is a sub-class of y if and only if $\forall z(z \in x \Rightarrow z \in y)$. In symbols $x \subseteq y$.

If $y \subseteq x$ and x is a set, we want also that y be a set.

Power-set Axiom

If x is a set, then there is a set y such that $\forall z\, (z \subseteq x \Leftrightarrow z \in y)$.

The set y is unique by extensionality, and is denoted by $\mathscr{P}(x)$.

Corollary 16.2

If B is a set and $A \subseteq B$ then A is a set. Equivalently: if A is a proper class and $A \subseteq B$ then B is a proper class.

The empty class $\{x \mid x \neq x\} = \emptyset$ is included in any set, so it is a set.

Pairing Axiom

$\mathrm{Set}(x) \wedge \mathrm{Set}(y) \Rightarrow \exists z\,(\mathrm{Set}(z) \wedge \forall w\,(w \in z \Leftrightarrow w = x \vee w = y))$.

The set z above is denoted by $\{x, y\}$. If $x = y$ we write $\{x\}$. Define

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Proposition 16.3

For all sets x, y, z, and w,
$(x, y) = (z, w) \Leftrightarrow x = z \wedge y = w$.

See the textbook for the proof.

Foundation Axiom

$x \neq \emptyset \Rightarrow \exists y (y \in x \wedge x \cap y = \emptyset)$.

If $x \in x$, then $\{x\}$ is a set, and by the axiom of foundation there is $y \in \{x\}$ such that $y \cap \{x\} = \emptyset$: but $y = x$ and $x \in x \cap \{x\}$, a contradiction.

Definition

$\mathrm{V} \stackrel{\text{def}}{=} \{x \mid x = x\}$ is the class of all sets.

Definition

If A is a class the **union** of A is $\bigcup A = \bigcup_{x \in A} x \stackrel{\text{def}}{=} \{y \mid \exists x \in A (y \in x)\}$ and the **intersection** of A is $\bigcap A = \bigcap_{x \in A} x \stackrel{\text{def}}{=} \{y \mid \forall x \in A (y \in x)\}$.

By convention: if $A = \emptyset$ then $\bigcap A = \emptyset$.

Union Axiom

If A is a set, then also $\bigcup A$ is a set.

If x and y are sets, then $\{x, y\}$ and $x \cup y \stackrel{\text{def}}{=} \bigcup\{x, y\}$ are sets.
$A \times B \stackrel{\text{def}}{=} \{(x, y) \mid x \in A, y \in B\} = \{c \mid \exists a \exists b(a \in A \wedge b \in B \wedge c = (a, b))\}$ is a class, and it exists by comprehension.

Proposition 16.6

If A and B are sets, then also $A \times B$ is a set.

Proof.

It is enough to find a set containing $A \times B$. If $x \in A$ and $y \in B$, then $\{x\}, \{x, y\} \subseteq A \cup B$ and therefore $(x, y) = \{\{x\}, \{x, y\}\} \subseteq \mathscr{P}(A \cup B)$. It follows that $A \times B \subseteq \mathscr{P}(\mathscr{P}(A \cup B))$. □

By the pairing and union axioms we can construct infinitely many new sets: $\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\{\{\emptyset\}\}\}\}, \ldots$ or
$\{\emptyset\} = \mathbf{S}(\emptyset), \{\emptyset, \{\emptyset\}\} = \mathbf{S}(\{\emptyset\}), \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \mathbf{S}(\{\emptyset, \{\emptyset\}\}), \ldots$ where

$$\mathbf{S}(x) \stackrel{\text{def}}{=} x \cup \{x\}$$

is the **successor** x.
A class I is **inductive** if $\emptyset \in I \wedge \forall x(x \in I \Rightarrow \mathbf{S}(x) \in I)$.

Axiom of Infinity

There is an inductive *set*.

Definition

$\mathbb{N} \stackrel{\text{def}}{=} \bigcap \mathcal{I}$ where $\mathcal{I}$ is the class of all inductive sets.
$0 = \emptyset$, $1 = \mathbf{S}(0)$, $2 = \mathbf{S}(1) = \mathbf{S}(\mathbf{S}(0))$, $\ldots$

Proposition 16.7

$\mathbb{N} \in \mathcal{I}$ and $\forall n \in \mathbb{N}\,(n = 0 \vee \exists m \in \mathbb{N}\,(n = \mathbf{S}(m)))$.

Proof.

$0 \in I$ for all $I \in \mathcal{I}$, and hence $0 \in \bigcap \mathcal{I} = \mathbb{N}$.
Fix $n \in \mathbb{N}$: $n \in I$ and hence $\mathbf{S}(n) \in I$ for all $I \in \mathcal{I}$. Being $I \in \mathcal{I}$ arbitrary, then $\mathbf{S}(n) \in \bigcap \mathcal{I} = \mathbb{N}$. Thus $\mathbb{N} \in \mathcal{I}$.
Let $n \in \mathbb{N} \setminus \{0\}$ and towards a contradiction suppose that $n \neq \mathbf{S}(m)$ for all $m \in \mathbb{N}$. Then the set $J = \mathbb{N} \setminus \{n\}$ satisfies the formula defining $\mathcal{I}$ and hence $J \in \mathcal{I}$. It follows that $J \supseteq \bigcap \mathcal{I} = \mathbb{N}$, but $J \subset \mathbb{N}$ by construction: a contradiction. □

Proposition 16.8—Principle of Induction on $\mathbb{N}$

Suppose that $0 \in I \subseteq \mathbb{N}$ and $\forall n\,(n \in I \Rightarrow \mathbf{S}(n) \in I)$. Then $I = \mathbb{N}$.

Proof.

$I \in \mathcal{I}$, and so $I \supseteq \mathbb{N}$. □

A **binary relation** is a class of ordered pairs. A binary relation F is **functional** if $(x, y), (x, y') \in F \Rightarrow y = y'$.
We write $x\ R\ y$ rather than $(x, y) \in R$. If R is functional, then $R(x) =$ the unique y (if it exists) such that $(x, y) \in R$.

$$\begin{aligned} \operatorname{dom}(R) &= \{x \mid \exists y((x, y) \in R)\} \\ \operatorname{ran}(R) &= \{y \mid \exists x((x, y) \in R)\} \\ \operatorname{fld}(R) &= \operatorname{dom}(R) \cup \operatorname{ran}(R). \end{aligned}$$

Proposition 16.9

If R is a set, then $\mathrm{dom}(R)$, $\mathrm{ran}(R)$, $\mathrm{fld}(R)$ are sets.

Proof.

If $x \in \mathrm{dom}(R)$ then $x \in \{x\} \in (x, y) \in R$, for some y, and so $x \in \bigcup(\bigcup R)$, hence $\mathrm{dom}(R) \subseteq \bigcup\bigcup R$. The case for $\mathrm{ran}(R)$ and $\mathrm{fld}(R)$ are similar. □

Proposition 16.10

Let $\mathcal{F}$ be a class of functions and suppose it is upward-directed by $\subseteq$. Then $\bigcup \mathcal{F}$ is a functional relation.

Proof.

$\bigcup \mathcal{F}$ is a class of ordered pairs. Suppose $(x, y) \in \bigcup \mathcal{F}$ and $(x, z) \in \bigcup \mathcal{F}$ and hence $(x, y) \in f$ e $(x, z) \in g$, for some $f, g \in \mathcal{F}$. Let $h \in \mathcal{F}$ be such that $f, g \subseteq h$: then $(x, y), (x, z) \in h$ and hence $y = z$. □

Theorem 16.11

There is no function f such that $\mathrm{dom}(f) = \mathbb{N}$ and $f(\mathbf{S}(n)) \in f(n)$ for all $n \in \mathbb{N}$

Proof.

Suppose such an f exists. As $\emptyset \neq \mathrm{ran}(f)$, by the axiom of foundation there is $y \in \mathrm{ran}(f)$ such that $y \cap \mathrm{ran}(f) = \emptyset$. Let $n \in \mathbb{N}$ be such that $y = f(n)$. But $f(\mathbf{S}(n)) \in f(n) \cap \mathrm{ran}(f)$: a contradiction. □

Definition

If F is a functional relation and A is a class, let

$$F[A] = \{F(x) \mid x \in A \cap \mathrm{dom}(F)\}$$
$$F \restriction A = \{(x, y) \in F \mid x \in A\}.$$

Exercise

If F is a set, then so is $F[A]$.

Axiom of Replacement (strong form)

If F is a functional relation and A is a set, then $F[A]$ is a set.

${}^{A}B = B^{A} = \{F \mid F \colon A \to B\}$.

Proposition 16.13

If A and B are sets, then B^{A} is a set.

Proof.

$B^{A} \subseteq \mathscr{P}(A \times B)$. □

From the axiom of replacement we obtain:

Proposition 16.14

If A is a proper class and $A \precsim B$, then B is a proper class.

Notation

$\langle a_i \mid i \in I \rangle$ is the function $I \ni i \mapsto a_i$.

For example, $s = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ is the function with domain $n = \{0, 1, \ldots, n-1\}$ assigning to every $i < n$ the set a_i.
$n = \mathrm{dom}(s)$ is the **length** of s, and it is denoted with $\mathrm{lh}(s)$.

If X is a class, then $X^{<\mathbb{N}} = \{s \mid s \text{ is a finite string and } \operatorname{ran}(s) \subseteq X\}$.

Exercise

Show that if X is a set, then $X^{<\mathbb{N}} = \bigcup\{X^n \mid n \in \mathbb{N}\}$ is a set.

If I is a set and $\langle A_i \mid i \in I\rangle$ is a sequence of sets, let
$\bigtimes_{i \in I} A_i = \{f \mid f \text{ is a function, } \operatorname{dom}(f) = I \text{ and } \forall i \in I\,(f(i) \in A_i)\}$.
Thus if $A_i = A$ for each $i \in I$, then $\bigtimes_{i \in I} A_i = A^I$.
If $A_{i_0} = \emptyset$ for some $i_0 \in I$ then $\bigtimes_{i \in I} A_i = \emptyset$. Does the vice-versa hold? We want to exchange the quantifiers, from '$\forall i \in I \exists x(x \in A_i)$' to '$\exists f \forall i \in I(f(i) \in A_i)$'.

Axiom of Choice (AC)

If $\mathcal{A} \neq \emptyset$ is a set and $\forall A \in \mathcal{A}\,(A \neq \emptyset)$, then there is $f \colon \mathcal{A} \to \bigcup \mathcal{A}$ such that $\forall A \in \mathcal{A}\,(f(A) \in A)$.

The axioms of MK

Recapping: MKC = MK + AC, and the axioms of MK are:

- Extensionality: $\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y$
- Comprehension: $\exists A \forall x\,(x \in A \Leftrightarrow \exists z(x \in z) \wedge \varphi(x, y_1, \ldots, y_n))$, where x is free in $\varphi(x, y_1, \ldots, y_n)$ and A different from $x, y_1, \ldots, y_n$
- Set-existence: $\exists x \exists y\,(x \in y)$
- Power-set: $\mathrm{Set}(x) \Rightarrow \exists z\,(\mathrm{Set}(z) \wedge \forall t\,(t \in z \Leftrightarrow t \subseteq x))$
- Pairing: $\mathrm{Set}(x) \wedge \mathrm{Set}(y) \Rightarrow \exists z\,(\mathrm{Set}(z) \wedge \forall w\,(w \in z \Leftrightarrow w = x \vee w = y))$
- Foundation: $x \neq \emptyset \Rightarrow \exists y\,(y \in x \wedge y \cap x = \emptyset)$
- Union: $\mathrm{Set}(x) \Rightarrow \exists u\,(\mathrm{Set}(u) \wedge \forall z\,(z \in u \Leftrightarrow \exists y(y \in x \wedge z \in y)))$
- Infinity: $\exists I\,(\mathrm{Set}(I) \wedge \emptyset \in I \wedge \forall x\,(x \in I \Rightarrow \mathbf{S}(x) \in I))$
- Replacement: $\forall F \forall A\,\big(\big(\forall x \exists! y (x, y) \in F \wedge \mathrm{Set}(A)\big) \Rightarrow \mathrm{Set}(F[A])\big)$.

The axioms of ZF

The theory Zermelo-Frænkel ZF is formulated in the language with just one binary predicate $\in$. The objects in this theory are *sets*. The axioms of ZF are:

- Extensionality: $\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y$
- Separation: $\exists A \forall x\,(x \in A \Leftrightarrow x \in B \wedge \varphi(x, y_1, \ldots, y_n, B))$, where x is free in $\varphi(x, y_1, \ldots, y_n, B)$ and A differs from $x, y_1, \ldots, y_n, B$. $A = \{x \in B \mid \varphi(x, y_1, \ldots, y_n, B)\}$ is the subset of B made-up of the elements that enjoy property φ.
- Power-set: $\exists z \forall t\,(t \in z \Leftrightarrow t \subseteq x)$
- Pairing: $\exists z \forall w\,(w \in z \Leftrightarrow w = x \vee w = y)$
- Foundation: $x \neq \emptyset \Rightarrow \exists y\,(y \in x \wedge y \cap x = \emptyset)$
- Union: $\exists u \forall z\,(z \in u \Leftrightarrow \exists y(y \in x \wedge z \in y))$
- Infinity: $\exists I\,(\emptyset \in I \wedge \forall x\,(x \in I \Rightarrow \mathbf{S}(x) \in I))$
- Replacement: $\forall x \in A \exists! y \varphi \Rightarrow \exists B \forall x \in A \exists y \in B \varphi$, for any formula $\varphi(x, y, A, w_1, \ldots, w_n)$.

Definition.

A finitary function or **operation** on X is an $f\colon X^n \to X$ for some $n \in \mathbb{N}$, called arity of f, $n = \operatorname{ar}(f)$.

If $n = 0$ then $f\colon \{\emptyset\} \to X$, so the 0-ary functions on X can be identified with the elements of X.
$Y \subseteq X$ is closed under f if $f[Y^n] \subseteq Y$.

Exercise

Let $Y \subseteq X$ and let $\mathcal{C} = \{Z \subseteq X \mid Y \subseteq Z \wedge Z \text{ closed under } f\}$. Show that $\mathcal{C} \neq \emptyset$ and that $\bigcap \mathcal{C}$ is the smallest $Z \subseteq X$ containing Y and closed under f.

The set $\bigcap \mathcal{C}$ is the **closure** of Y under f, in symbols $\operatorname{Cl}_f(Y)$.
Similarly one defines $\operatorname{Cl}_{\mathcal{F}}(Y)$ when $\mathcal{F}$ is a family of finitary functions on X.