Elements of Mathematical Logic

Section 18 of Chapter V

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Definitions 18.1 and 18.2

Let X be a class and let $R \subseteq X \times X$ be a relation on X.

- R is **regular** or **left-narrow** if $\{y \in X \mid y \mathrel{R} x\}$ is a set, for all $x \in X$.
- R is **well-founded** if every nonempty subclass of X contains an R-minimal element that is

$$\forall Y \subseteq X \, (Y \neq \emptyset \Rightarrow \exists y \in Y \, \forall z \in Y \, (z \neq y \Rightarrow (z, y) \notin R)).$$

  Otherwise R is **ill-founded**.
- R is a **well-order** on X if it is a linear order, left-narrow, and well-founded.

The axiom of foundation implies that the membership relation $\{(x, y) \in \mathrm{V} \mid x \in y\}$ is irreflexive and well-founded, and since $\{y \mid y \in x\} = x$ is a set for all $x \in \mathrm{V}$, it is also regular.

Theorem 18.3

If the set X is well-orderable, then there is a choice function on X.

Proof.

If $\emptyset \neq A \subseteq X$ let $f(A)$ be the minimum of A. □

The following result is a straightforward generalization of Propositions 13.3 and 13.4 and Corollaries 13.5 and 13.6.

Theorem 18.4

Let $\langle A, \leq \rangle$ be a well-ordered class.

- If $f \colon A \to A$ is increasing, then $\forall a \in A \, (a \leq f(a))$; if moreover f is bijective then $f = \mathrm{id}_A$.
- If $\langle A, \leq \rangle$ and $\langle B, \trianglelefteq \rangle$ are isomorphic well-ordered classes, then the isomorphism is unique.
- If $a \in A$, then $\langle A, \leq \rangle$ and $\langle \mathrm{pred}(a, A; \leq), \leq \rangle$ are not isomorphic.

If $f\colon A \to A$ is increasing, then $\forall a \in A\,(a \leq f(a))$; if moreover f is bijective then $f = \mathrm{id}_A$.

Towards a contradiction, let $\bar{a} \in A$ be least such that $f(\bar{a}) < \bar{a}$. By minimality $f(f(\bar{a})) \geq f(\bar{a})$, and $f(f(\bar{a})) < f(\bar{a})$ since f is increasing. $f(a) \geq a$ and $f^{-1}(a) \geq a$, so that $f(a) = a$.

If $\langle A, \leq \rangle$ and $\langle B, \trianglelefteq \rangle$ are isomorphic well-ordered classes, then the isomorphism is unique.

If $f, g\colon A \to B$ are isomorphisms, then $g^{-1} \circ f\colon A \to A$ is an isomorphism, so $g^{-1}(f(a)) = a$, that is $f(a) = g(a)$.

If $a \in A$, then $\langle A, \leq \rangle$ and $\langle \mathrm{pred}(a, A; \leq), \leq \rangle$ are not isomorphic.

If $f\colon A \to \mathrm{pred}(a, A; <)$ is an isomorphism, then $f\colon A \to A$ is increasing, so $\forall x \in A(x \leq f(x))$ and hence $a \leq f(a)$, a contradiction.

Definition 18.5

A class A is **transitive** if $\bigcup A \subseteq A$, that is $x \in a \in A \Rightarrow x \in A$.
An ordinal is a transitive set, whose elements are transitive.

Proposition 18.6

1. If x is a transitive set, then $\bigcup x$ and $\mathbf{S}(x)$ are transitive.
2. If $\alpha \in \mathrm{Ord}$ then $\alpha \subseteq \mathrm{Ord}$ and $\mathbf{S}(\alpha) \in \mathrm{Ord}$.
3. If x is a set of ordinals, then $\bigcup x \in \mathrm{Ord}$.

Proposition 18.7

Ord is a proper class.

Proof.

If $\alpha \in \mathrm{Ord}$ and $\beta \in \alpha$, then $\beta \in \mathrm{Ord}$, so Ord is a transitive class. If Ord were a set, then it would be an ordinal, and hence $\mathrm{Ord} \in \mathrm{Ord}$: a contradiction. □

Theorem 18.8

$\forall \alpha, \beta \in \mathrm{Ord}\,(\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$.

Proof.

We must prove that
$A = \{\alpha \in \mathrm{Ord} \mid \exists \beta \in \mathrm{Ord}\,(\alpha \notin \beta \wedge \alpha \neq \beta \wedge \beta \notin \alpha)\}$ is empty. If $A \neq \emptyset$, then there would be an $\bar{\alpha} \in A$ such that $\bar{\alpha} \cap A = \emptyset$. Then
$B = \{\beta \in \mathrm{Ord} \mid \beta \notin \bar{\alpha} \wedge \beta \neq \bar{\alpha} \ \wedge \ \bar{\alpha} \notin \beta\} \neq \emptyset$ so there is $\bar{\beta} \in B$ such that $\bar{\beta} \cap B = \emptyset$. If $\gamma \in \bar{\alpha}$ then $\gamma \notin A$, so in particular
$\bar{\beta} \in \gamma \vee \bar{\beta} = \gamma \vee \gamma \in \bar{\beta}$. The first two possibilities together with transitivity of $\bar{\alpha}$ imply that $\bar{\beta} \in \bar{\alpha}$, against $\bar{\beta} \in B$. Thus $\gamma \in \bar{\beta}$. Being γ arbitrary, we have that $\bar{\alpha} \subseteq \bar{\beta}$. Similarly $\bar{\beta} \subseteq \bar{\alpha}$ and thus $\bar{\alpha} = \bar{\beta}$: a contradiction. □

Corollary 18.9

$\in$ is a strict well-order on Ord, and thus on every ordinal α.

Write $\alpha < \beta$ and $\alpha \leq \beta$ for $\alpha \in \beta$ and $(\alpha \in \beta \vee \alpha = \beta)$.
If $\emptyset \neq A \subseteq \mathrm{Ord}$, the $\in$-minimal element of A is the minimum of A.
Every α yields a well-order $\langle \alpha, \in \rangle$ and if $\beta \in \alpha$, then $\beta = \mathrm{pred}(\beta, \alpha; \in)$, so $\langle \alpha, < \rangle \cong \langle \beta, < \rangle \Leftrightarrow \alpha = \beta$.

Proposition 18.10

1. Suppose $f \colon \alpha \to \beta$ is increasing. Then $\forall \gamma \in \alpha \, (\gamma \leq f(\gamma))$ and $\alpha \leq \beta$.
2. If $f \colon \alpha \to \beta$ is an isomorphism, then $\alpha = \beta$ and f is the identity.

(See textbook.)
Similarly, if $f \colon \mathrm{Ord} \to \mathrm{Ord}$ is increasing, then $\gamma \leq f(\gamma)$, and if f is moreover surjective then it is the identity.

Theorem 18.11

Every well-ordered *set* is isomorphic to an ordinal and every well-ordered *class* is isomorphic to Ord. Moreover the ordinal and the isomorphism are unique.

Proof.

Let $\langle X, < \rangle$ be a well-ordered set and let
$A = \{\alpha \in \mathrm{Ord} \mid \exists x \in X\, (\langle \alpha, \in \rangle \cong \langle \mathrm{pred}(x) \rangle)\}$. Suppose
$f \colon \langle \alpha, \in \rangle \to \langle \mathrm{pred}(x), < \rangle$ witnesses that $\alpha \in A$. If $\beta \in \alpha$ then
$f \restriction \beta \colon \langle \beta, \in \rangle \to \langle \mathrm{pred}(f(\beta)), < \rangle$ is an isomorphism so $\beta \in A$. It follows that A is transitive so it is an ordinal. Let $f \colon A \to X$ be the map assigning to every $\alpha \in A$ the unique $x \in X$ such that $\langle \alpha, \in \rangle \cong \langle \mathrm{pred}(x), < \rangle$. Thus $\mathrm{ran}(f)$ is an initial segment of X. If $\mathrm{ran}(f) \neq X$, then $\mathrm{ran}(f) = \mathrm{pred}(\bar{x})$, for some $\bar{x} \in X$ and the isomorphism $f \colon \langle A, \in \rangle \to \langle \mathrm{pred}(\bar{x}), < \rangle$ witnesses that the ordinal A belongs to the set A: a contradiction.
The case for proper classes is similar.
Uniqueness follows from the preceding results. □

Theorem 18.12

If $\langle A, <\rangle$ and $\langle B, \prec\rangle$ are well-ordered classes, then *exactly one* of the following holds:

1. $\exists a \in A\,(\langle \operatorname{pred} a, <\rangle \cong \langle B, \prec\rangle)$
2. $\exists b \in B\,(\langle \operatorname{pred} b, \prec\rangle \cong \langle A, <\rangle)$
3. $\langle A, <\rangle \cong \langle B, \prec\rangle$.

In particular, two well-ordered proper classes are isomorphic.

Proof

Let $F = \{(a, b) \in A \times B \mid \langle \operatorname{pred} a, < \rangle \cong \langle \operatorname{pred} b, \prec \rangle\}$. Then F is a functional relation. If $a \in \operatorname{dom} F$ and $a' < a$, then $(a', F(a')) \in F$; similarly, if $b' \prec b \in \operatorname{ran} F$ then there is $a' < a$ such that $F(a') = b'$. Therefore F is a functional relation from an initial segment of A onto an initial segment of B.

It is enough to show that $\operatorname{dom} F \neq A$ and $\operatorname{ran} F \neq B$ cannot both be true. Towards a contradiction, let $\bar{a}$ be least in $A \setminus \operatorname{dom} F$ and let $\bar{b}$ be least in $B \setminus \operatorname{ran} F$; then $F \colon \langle \operatorname{pred} \bar{a}, < \rangle \to \langle \operatorname{pred} \bar{b}, \prec \rangle$ is an isomorphism, so $(\bar{a}, \bar{b}) \in F$, a contradiction.

Proposition 18.13

If $A \neq \emptyset$ is a nonempty class of ordinals, then $\min A = \bigcap A$.

Proof.

Suppose $\emptyset \neq A \subseteq \mathrm{Ord}$ and let $\bar{\alpha} \in A$ be such that $\bar{\alpha} \cap A = \emptyset$. Then $\forall \alpha \in A\,(\bar{\alpha} \subseteq \alpha)$, so $\bigcap A = \bar{\alpha} = \min A$. □

Corollary 18.14

There is no infinite descending chain of ordinals, that is
$\neg \exists f\,(f \colon \mathbb{N} \to \mathrm{Ord} \wedge \forall n(f(\mathbf{S}(n)) < f(n)))$.

Lemma 18.15

1. Every natural number is an ordinal.
2. If $n \in \mathbb{N}$ and $x \in n$ then $x \in \mathbb{N}$.

Proof.

1. Towards a contradiction, suppose $X = \mathbb{N} \setminus \mathrm{Ord}$ is nonempty and let $n \in X$ be such that $n \cap X = \emptyset$. As 0 is an ordinal, it follows that $n \neq 0$ so $n = \mathbf{S}(m)$ for some $m \in \mathbb{N}$. As $m \in \mathrm{Ord}$ then $\mathbf{S}(m) \in \mathrm{Ord} \cap \mathbb{N}$: a contradiction.
2. Towards a contradiction, suppose $X = \{n \in \mathbb{N} \mid \exists x \in n\, (x \notin \mathbb{N})\}$ is nonempty and let $\bar{n} \in X$ be such that $\bar{n} \cap X = \emptyset$. Fix $\bar{x} \in \bar{n}$ such that $\bar{x} \in \bar{n} \setminus \mathbb{N}$. $\bar{n} = \mathbf{S}(\bar{m})$, for some $\bar{m} \in \mathbb{N}$, so either $\bar{x} \in \bar{m}$ or $\bar{x} = \bar{m}$. Both possibilities yield a contradiction. □

An ordinal α is a **successor** if $\alpha = \mathbf{S}(\beta)$ for some β. Clearly $\alpha < \mathbf{S}(\alpha)$ and there is no β such that $\alpha < \beta < \mathbf{S}(\alpha)$. If an ordinal is neither a successor nor is 0, then it is **limit**.

Theorem 18.16

$\mathbb{N}$ is the smallest limit ordinal.

Proof.

$\mathbb{N}$ is an ordinal and there are no limit ordinals less than $\mathbb{N}$. It is enough to check that $\mathbb{N}$ is not a successor. If, towards a contradiction, $\mathbb{N} = \mathbf{S}(\alpha)$, then $\alpha \in \mathbb{N}$, so $\mathbf{S}(\alpha) \in \mathbb{N}$, that is $\mathbb{N} \in \mathbb{N}$: a contradiction. □

Notation

$\mathbb{N} = \omega$.

Proposition 18.18

If A is a set of ordinals, then $\bigcup A = \sup A$.

Proof.

Let A be a set of ordinals. $\bigcup A$ is the smallest set containing every $\alpha \in A$. Since $\bigcup A$ is an ordinal and since $\subseteq$ agrees with $\leq$ on the ordinals, it follows that $\bigcup A = \sup A$. $\square$

Proposition 18.17

1. $\alpha < \beta \Leftrightarrow \alpha \subset \beta$;
2. $\alpha \leq \beta \Leftrightarrow \alpha \subseteq \beta$;
3. $\alpha < \beta \Leftrightarrow \mathbf{S}(\alpha) \leq \beta$;
4. $x \subseteq \alpha \Rightarrow (\bigcup x = \alpha \ \vee \ \bigcup x < \alpha)$;
5. $\bigcup(\mathbf{S}(\alpha)) = \alpha$;
6. $\alpha = \mathbf{S}(\bigcup \alpha) \vee \alpha = \bigcup \alpha$;
7. $\bigcup \alpha = \alpha \Leftrightarrow (\alpha = 0 \vee \alpha \text{ limit}) \Leftrightarrow \langle \alpha, < \rangle$ has no maximum.

Proof of Proposition 18.17

1 $\alpha < \beta \Leftrightarrow \alpha \subset \beta$

If $\alpha \in \beta$ then $\alpha \subseteq \beta$ by transitivity. Foundation implies that $\alpha \neq \beta$, so $\alpha \subset \beta$. Conversely suppose $\alpha \subset \beta$: by foundation $\beta \notin \alpha$ and since $\beta \neq \alpha$ it follows that $\alpha \in \beta$.

2 $\alpha \leq \beta \Leftrightarrow \alpha \subseteq \beta$

is similar to **1**

3 $\alpha < \beta \Leftrightarrow \mathbf{S}(\alpha) \leq \beta$

Let $\alpha < \beta$. As $\beta \in \mathbf{S}(\alpha)$ is impossible, then $\beta = \mathbf{S}(\alpha)$ or $\mathbf{S}(\alpha) \in \beta$. The converse implication is immediate.

(continues)

4 $x \subseteq \alpha \Rightarrow (\bigcup x = \alpha \ \vee \ \bigcup x < \alpha)$

$\bigcup x$ is an ordinal so it is comparable with α. But $\alpha \in \bigcup x$ implies that $\alpha \in \beta \in x \subseteq \alpha$, for some β: a contradiction. So $\bigcup x \leq \alpha$.

5 $\bigcup(\mathbf{S}(\alpha)) = \alpha$

$\beta \in \bigcup \mathbf{S}(\alpha)$ if and only if $\beta \in \gamma \in \alpha$ for some γ or $\beta \in \alpha$. So $\beta \in \bigcup \mathbf{S}(\alpha) \Leftrightarrow \beta \in \alpha$.

6 $\alpha = \mathbf{S}(\bigcup \alpha) \vee \alpha = \bigcup \alpha$

From **5** we get $\bigcup \alpha \leq \alpha$. If $\bigcup \alpha < \alpha$, then by **3** $\mathbf{S}(\bigcup \alpha) \leq \alpha$, so it is enough to show that the strict inequality does not hold. Assume $\mathbf{S}(\bigcup \alpha) \in \alpha$; as $\bigcup \alpha \in \mathbf{S}(\bigcup \alpha)$ then $\bigcup \alpha \in \bigcup \alpha$: a contradiction.

7 $\bigcup \alpha = \alpha \Leftrightarrow (\alpha = 0 \vee \alpha$ limit$) \Leftrightarrow \langle \alpha, < \rangle$ has no maximum

Follows from **5** and **6** .

The topology on the ordinals

Any ordinal can be seen as a topological space, α is a subspace of β when $\alpha < \beta$, so we the topology on α is induced by the interval topology on $\langle \mathrm{Ord}, \leq \rangle$. The problem is that *it does not make sense* speak of a topology on a proper class!

Definition 21.21

Let $\Omega \leq \mathrm{Ord}$. A class $A \subseteq \Omega$ is **open** in Ω if for all $\alpha \in A$ there is an interval I such that $\alpha \in I \subseteq A$. A class $C \subseteq \Omega$ is **closed** in Ω if $\Omega \setminus C$ is open in Ω; equivalently: $\forall \lambda \big(0 < \bigcup (C \cap \lambda) = \lambda \Rightarrow \lambda \in C\big)$.

Which conditions must $f \colon \Omega \to \mathrm{Ord}$ satisfy in order to be continuous? Continuity is never a problem on successor ordinals, since they are isolated points. Suppose then $\gamma < \Omega$ is limit. If $f(\gamma)$ is a successor, then f is eventually constant below γ.

The topology on the ordinals

If $f(\gamma)$ is limit, the for every $\delta < f(\gamma)$ there is a $\beta < \gamma$ such that $[\beta; \gamma]$ is mapped by f into $[\delta; f(\gamma)]$.
Suppose $f \colon \Omega \to \mathrm{Ord}$ in monotone. Then f is continuous if and only if

$$\forall\lambda\bigl(\lambda \text{ limit} \Rightarrow f(\lambda) = \sup\nolimits_{\beta<\lambda} f(\beta)\bigr)$$

and if λ is limit,

$$X \subseteq \lambda \wedge \sup X = \lambda \Rightarrow f(\lambda) = \sup_{\nu \in X} f(\nu).$$

Moreover, if f is increasing, then $f(\lambda)$ is a limit ordinal.

Cardinals

x is **finite** if it is in bijection with a natural number, otherwise it is is **infinite**.

A **cardinal** is an ordinal κ which is not in bijection with any $\alpha < \kappa$.

Cardinals are usually denoted with greek letters such as κ, λ, ... and Card is the class of all cardinals.

A class X is **well-orderable** if there is a well-order on X — equivalently by Theorem 18.11 if X is in bijection with some $\Omega \leq \mathrm{Ord}$. If X is well-orderable and Y is in bijection with (or even just: the surjective image of) X, then Y is well-orderable; conversely, if Y is well-orderable and $X \precsim Y$, then X is well-orderable.

Definition 18.20

If X is a well-orderable *set*, its **cardinality** is the smallest ordinal $|X|$ in bijection with X.

In particular $|\alpha|$, is the smallest ordinal $\beta \asymp \alpha$ and hence $|\alpha| \leq \alpha$.

Proposition 18.21

If $\kappa, \lambda \in \mathrm{Card}$

1. $\kappa = \lambda$ if and only if $\kappa \asymp \lambda$,
2. $\kappa \leq \lambda$ if and only if $\kappa \precsim \lambda$.

Proof.

1. Suppose that $\kappa \asymp \lambda$ and that $\kappa \neq \lambda$, for example $\kappa < \lambda$. Then λ would be in bijection with a strictly smaller ordinal: a contradiction.
2. Suppose $f \colon \kappa \rightarrowtail \lambda$. If, towards a contradiction, $\lambda < \kappa$, then let $j \colon \lambda \to \kappa$ be the identity function. By the Cantor-Schröder-Bernstein Theorem $\kappa \asymp \lambda$ so $\kappa = \lambda$ by 1 : a contradiction. □

Proposition 18.22

1. If $\alpha \geq \omega$ then $|\alpha| = |\mathbf{S}(\alpha)|$,
2. $|\alpha| \leq \beta \leq \alpha \Rightarrow |\alpha| = |\beta|$,
3. $|\alpha| = |\beta|$ if and only if $\alpha \asymp \beta$,
4. $|\alpha| \leq |\beta|$ if and only if $\alpha \precsim \beta$.

Proof.

1. $f \colon \mathbf{S}(\alpha) \to \alpha, \quad f(\beta) = \begin{cases} \mathbf{S}(\beta) & \text{if } \beta < \omega, \\ \beta & \text{if } \omega \leq \beta < \alpha, \\ 0 & \text{if } \beta = \alpha, \end{cases}$ is a bijection.

2. Let $f \colon \alpha \to |\alpha|$ be bijective. As $f \colon \alpha \to \beta$ is injective and $\beta \precsim \alpha$, then $|\alpha| = |\beta|$ by the Cantor-Schröder-Bernstein Theorem and preceding Proposition.

3. and 4. follow from the previous Proposition. □

Given a set X consider the class $A = \{(\alpha, f) \mid \alpha \in \mathrm{Ord} \wedge f \colon \alpha \rightarrowtail X\}$. To $(\alpha, f) \in A$ associate the well-order $W_{(\alpha,f)}$ on $\mathrm{ran}(f) \subseteq X$ induced by f: $x \, W_{(\alpha,f)} \, y \Leftrightarrow f^{-1}(x) < f^{-1}(y)$.
Then $f \colon \langle \alpha, < \rangle \to \left\langle \mathrm{ran}(f), W_{(\alpha,f)} \right\rangle$ is an isomorphism.
If $(\alpha, f), (\beta, g) \in A$ and $W_{(\alpha,f)} = W_{(\beta,g)}$ then $g^{-1} \circ f \colon \langle \alpha, < \rangle \to \langle \beta, < \rangle$ is an isomorphism, so $\alpha = \beta$ and $f = g$. Thus $A \to \mathscr{P}(X \times X)$, $(\alpha, f) \mapsto W_{(\alpha,f)}$ is injective, so $A \precsim \mathscr{P}(X \times X)$ and hence A a set. Also

$$\mathrm{Hrtg}(X) = \{\alpha \in \mathrm{Ord} \mid \alpha \precsim X\} = \{\alpha \in \mathrm{Ord} \mid \exists f \; (\alpha, f) \in A\}$$

is a set. It is the smallest ordinal that does not inject into X.
$\mathscr{P}(X \times X) \twoheadrightarrow A \twoheadrightarrow \mathrm{Hrtg}(X)$. If $|\mathrm{Hrtg}(X)| < \mathrm{Hrtg}(X)$, then $\mathrm{Hrtg}(X) \rightarrowtail |\mathrm{Hrtg}(X)| \in \mathrm{Hrtg}(X)$ and hence $\mathrm{Hrtg}(X) \rightarrowtail X$: a contradiction.

Theorem 18.23

$\mathrm{Hrtg}(X)$ is the smallest ordinal that does not inject into X, and it is a cardinal. Moreover $\mathscr{P}(X \times X) \twoheadrightarrow \mathrm{Hrtg}(X)$.

$\mathrm{Hrtg}(\alpha)$ is denoted by α^+ and ω^+ is denoted by ω_1.

Theorem 18.25

If X is a set of cardinals, then $\sup X$ is a cardinal.

Proof.

If, towards a contradiction, $\lambda = \bigcup X \notin \mathrm{Card}$ then $|\lambda| < \lambda$ so that $|\lambda| < \kappa \leq \lambda$ for some $\kappa \in X$ and therefore $|\lambda| = |\kappa|$, that is κ would not be a cardinal: a contradiction. □

Corollary 18.26

Card is a proper class.

Cardinal addition and multiplication

Definition 18.27

Cardinal addition and **multiplication** are the binary operations $\mathrm{Card} \times \mathrm{Card} \to \mathrm{Card}$ defined by

$$\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}| \qquad \kappa \cdot \lambda = |\kappa \times \lambda|.$$

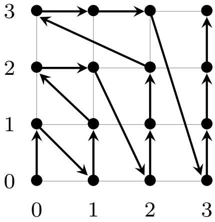
Note that $2 \leq \kappa, \lambda \Rightarrow \kappa + \lambda \leq \kappa \cdot \lambda$ and that this formula is true even when one of the two cardinals is 1 and the other is $\geq \omega$. Summarizing: if κ and λ are cardinals and if either $2 \leq \min(\kappa, \lambda)$ or else $1 = \min(\kappa, \lambda)$ and $\omega \leq \max(\kappa, \lambda)$, then

$$\max(\kappa, \lambda) \leq \kappa + \lambda \leq \kappa \cdot \lambda \leq \max(\kappa, \lambda) \cdot \max(\kappa, \lambda).$$

The **Gödel well-order on** $\mathrm{Ord} \times \mathrm{Ord}$ is defined by

$$(\alpha, \beta) <_{\mathrm{G}} (\gamma, \delta) \Leftrightarrow$$
$$\Big[\max(\alpha, \beta) < \max(\gamma, \delta) \vee \big(\max(\alpha, \beta) = \max(\gamma, \delta) \wedge (\alpha, \beta) <_{\mathrm{lex}} (\gamma, \delta)\big)\Big].$$

The ordering $<_{\mathrm{G}}$ agrees with the ordering on ω given by the enumeration



and if $\alpha < \beta$ then $\alpha \times \alpha$ is an initial segment of $\beta \times \beta$.

Theorem 18.28

Let κ be an infinite cardinal. Then $\mathrm{ot}(\kappa \times \kappa, <_{\mathrm{G}}) = \kappa$ and $|\kappa \times \kappa| = \kappa$.

Proof.

The function $\langle \kappa, < \rangle \to \langle \kappa \times \kappa, <_{\mathrm{G}} \rangle$, $\alpha \mapsto (\alpha, 0)$, is increasing so that $\kappa \leq \mathrm{ot}(\kappa \times \kappa, <_{\mathrm{G}})$. Therefore it is enough to show by induction on $\kappa \geq \omega$ that $\mathrm{ot}(\kappa \times \kappa, <_{\mathrm{G}}) \leq \kappa$, so that $|\kappa \times \kappa| = \kappa$.
Let $\alpha < \kappa$. If $\alpha < \omega$, then $|\alpha \times \alpha| < \omega$ by Proposition 13.20. If instead $\omega \leq \alpha$, then $\omega \leq |\alpha| < \kappa$ so by inductive assumption $|\alpha| \times |\alpha|$ is of cardinality $|\alpha|$. As $|\alpha| \times |\alpha| \asymp \alpha \times \alpha$, then $|\alpha \times \alpha| < \kappa$. Therefore we have shown that $\forall \alpha < \kappa \, (|\alpha \times \alpha| < \kappa)$. Fix $\alpha, \beta < \kappa$. The set $\mathrm{pred}(\alpha, \beta)$ of $<_{\mathrm{G}}$-predecessors of (α, β) is included in $\nu \times \nu$, where $\nu = \max\{\alpha, \beta\} + 1$, so $|\mathrm{pred}(\alpha, \beta)| \leq |\nu \times \nu| < \kappa$. Therefore we have shown that $\forall \alpha, \beta < \kappa \, (\mathrm{ot}\,\mathrm{pred}(\alpha, \beta) < \kappa)$ and hence $\mathrm{ot}(\kappa \times \kappa, <_{\mathrm{G}}) \leq \kappa$. □

Corollary 18.29

If κ and λ are cardinals different from 0 and at least one among κ and λ is infinite, then

$$\max(\kappa, \lambda) = \kappa + \lambda = \kappa \cdot \lambda.$$

Proposition 18.30

If $2 \leq \kappa \leq \lambda$ and λ is an infinite cardinal, then ${}^{\lambda}2 \asymp {}^{\lambda}\kappa \asymp {}^{\lambda}\lambda$.

Proof.

${}^{\lambda}2 \subseteq {}^{\lambda}\kappa \subseteq {}^{\lambda}\lambda \subseteq \mathscr{P}(\lambda \times \lambda) \asymp \mathscr{P}(\lambda) \asymp {}^{\lambda}2.$ □

Theorem 18.31

Let X be an infinite set such that $X \times X \asymp X$. Then $\forall n > 0\, ({}^{n}X \asymp X)$.
Moreover, $\omega \precsim X$ implies ${}^{<\omega}X \asymp X$.
In particular, if X is well-orderable and infinite, then $|{}^{<\omega}X| = |X|$.

Proof

Let $f \colon X \times X \to X$ be a bijection. Define by recursion on $n \geq 1$ bijections $j_n \colon {}^n X \to X$ as follows. Let $j_1(\langle x \rangle) = x$ for all $x \in X$, and since the function ${}^{n+1}X \to {}^n X \times X$, $s \mapsto (s \restriction n, s(n))$, is a bijection, it is possible to define j_{n+1} as

$$s \mapsto (s \restriction n, s(n)) \mapsto (j_n(s \restriction n), s(n)) \mapsto f(j_n(s \restriction n), s(n)).$$

Therefore ${}^n X \asymp X$ for all $n > 0$. Moreover given $\bar{x} \in X$ the function $j_\omega \colon {}^{<\omega} X \to \omega \times X$

$$j_\omega(s) = \begin{cases} (0, \bar{x}) & \text{if } s = \emptyset, \\ (n, j_n(s)) & \text{if } \mathrm{lh}(s) = n > 0, \end{cases}$$

is injective.

Definition 18.32

If $\langle X, \lhd \rangle$ is a well-ordered set and $\alpha \in \mathrm{Ord}$, let

$$[X]^{\alpha} = \{Y \subseteq X \mid \mathrm{ot}\langle Y, \lhd \rangle = \alpha\}.$$

Replacing $=$ with $\leq$ and $<$ in the formula above, the definition of $[X]^{\leq \alpha}$ and $[X]^{<\alpha}$ is obtained.

Every $x \in [\kappa]^n$ can be written as $x = \{\alpha_0, \ldots, \alpha_{n-1}\}$ with $\alpha_0 < \cdots < \alpha_{n-1} < \kappa$, and therefore it can be identified with the sequence $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in {}^n\kappa$. Such identification yields an injection $[\kappa]^n \rightarrowtail {}^n\kappa$ that extends to $[\kappa]^{<\omega} \rightarrowtail {}^{<\omega}\kappa = \bigcup_n {}^n\kappa$. Therefore for $n > 0$

$$\kappa \leq \left|[\kappa]^n\right| \leq \left|[\kappa]^{<\omega}\right| \leq \left|{}^{<\omega}\kappa\right| = \kappa$$

that is $\kappa = \left|[\kappa]^n\right| = \left|[\kappa]^{<\omega}\right|$.

Corollary 18.33

If X is infinite and well-orderable, then also $[X]^n$ and $[X]^{<\omega}$ are well-orderable, and $|[X]^n| = |[X]^{<\omega}| = |X|$ if $n > 0$.

Proposition 18.34

${}^{<\omega}X \asymp \omega \times {}^{<\omega}X \asymp {}^{<\omega}({}^{<\omega}X)$ for every set X.

Proof.

If X is empty or a singleton the result is trivial, so assume that X has at least two distinct elements x_0, x_1. Given $s \in {}^{<\omega}X$ define $s' \in {}^{<\omega}X$ as follows: if $s = \emptyset$ set $s' = \emptyset$, and if $s \neq \emptyset$ let $s' = x_0^{(\mathrm{lh}\, s)} {}^\frown \langle x_1 \rangle {}^\frown s$. Thus

$${}^{<\omega}({}^{<\omega}X) \to {}^{<\omega}X, \quad \langle s_0, \dots, s_n \rangle \mapsto s_0' {}^\frown s_1' {}^\frown \dots {}^\frown s_n',$$

is injective. Since $\omega \times {}^{<\omega}X \to {}^{<\omega}({}^{<\omega}X), \quad (n, s) \mapsto s^{(n)}$ is injective and ${}^{<\omega}X \precsim \omega \times {}^{<\omega}X$, the result follows from the Cantor-Schröder-Bernstein Theorem. □

Corollary 18.35

For any set X there is a set Y such that $X \precsim Y$ and such that $Y \asymp {}^{<\omega}Y$ and hence $Y \asymp Y \times Y$.

Proof.

Take $Y = {}^{<\omega}X$. □

Applications: vector spaces

Suppose V is a non-trivial vector space on a field $\mathbb{k}$. If V is well-orderable, then $\mathbb{k}$ is also well-orderable, and V has a **basis**. Conversely, if $\mathbb{k}$ is well-orderable and V has a well-orderable base, then V is well-orderable. To see this suppose $|\mathbb{k}| = \kappa$ and that $\{\mathbf{e}_\alpha \mid \alpha \in \lambda\}$ is a basis of V, where λ is a cardinal. If $\lambda < \omega$ then V is a finite dimensional vector space over a finite fields, so it is well-orderable and of size $\kappa^\lambda < \omega$. So we may assume that $\max(\kappa, \lambda) \geq \omega$. Thus for every $\mathbf{v} \in V$ there is a unique finite set $I = I(\mathbf{v}) = \{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq \lambda$ and a unique sequence of non-zero scalars $s = s(\mathbf{v}) \in {}^n\mathbb{k} \setminus \{0_\mathbb{k}\}$ such that

$$\mathbf{v} = \sum_{i<n} s(i)\mathbf{e}_{\alpha_i}.$$

When $\mathbf{v} = \mathbf{0}$ then $I(\mathbf{v}) = s(\mathbf{v}) = \emptyset$. If $\lambda < \omega$, then $V \asymp \mathbb{k}^\lambda$, hence

$$|V| = |{}^\lambda\kappa| = \begin{cases} \kappa & \text{if } \kappa \geq \omega, \\ \kappa^\lambda & \text{otherwise.} \end{cases}$$

If $\lambda \geq \omega$, then

$$V \to \bigcup_{n\in\omega} [\lambda]^n \times \bigcup_{n\in\omega} {}^n(\Bbbk \setminus \{0_\Bbbk\}), \quad \mathbf{v} \mapsto (I(\mathbf{v}), s(\mathbf{v}))$$

is injective, so $|V| = \max(\lambda, \kappa)$. Therefore

$$|V| = \begin{cases} \kappa^\lambda & \text{if } \kappa, \lambda < \omega, \\ \kappa & \text{if } \lambda < \omega \leq \kappa, \\ \max(\kappa, \lambda) & \text{if } \omega \leq \kappa, \lambda. \end{cases}$$

Suppose $\{\mathbf{e}_\alpha \mid \alpha \in \lambda\}$ and $\{\mathbf{e}'_\alpha \mid \alpha \in \lambda'\}$ are bases of V, with λ, λ' cardinals. If $\lambda < \omega$, then $\lambda = \lambda'$ by the Gramm-Shmidt algorithm; if $\omega \leq \lambda < \lambda'$, choose a finite set $I_\alpha \subseteq \lambda'$ for each $\alpha < \lambda$ so that $\mathbf{e}_\alpha$ is in the span of $\{\mathbf{e}'_\beta \mid \beta \in I_\alpha\}$, and hence $I = \bigcup_{\alpha<\lambda} I_\alpha$ is of size λ and $\{\mathbf{e}'_\alpha \mid \alpha \in I\}$ generates V, contradicting the assumption that $\{\mathbf{e}'_\alpha \mid \alpha \in \lambda'\}$ is a base. Therefore if V is well-orderable two bases have the same size, and the cardinality of any such base is called the **dimension** of V, in symbols $\dim(V)$.

Corollary 18.36

If V, W are well-orderable vector spaces over a well-orderable field $\mathbb{k}$, and $|V|, |W| > |\mathbb{k}|$, then

$$V \cong W \Leftrightarrow \dim(V) = \dim(W) \Leftrightarrow |V| = |W|.$$

Applications: free groups

$\boldsymbol{F}(X)$ is the set of all $\langle x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n} \rangle$ with $x_i \in X$ and $\varepsilon_i \in \{-1, 1\}$, with the proviso that if $x_i = x_{i+1}$ then $\varepsilon_i = \varepsilon_{i+1}$. Thus $\boldsymbol{F}(X)$ can be identified with a subset of $(\{1, -1\} \times X)^{<\omega}$, while X is identified with a subset of $\boldsymbol{F}(X)$ via $x \mapsto (1, x)$. If X is well-orderable, so is $\boldsymbol{F}(X)$, and if $|X| = \kappa \geq \omega$, then $|\boldsymbol{F}(X)| = \kappa$. Any function $f\colon X \to G$ can be uniquely extended to a homomorphism $\hat{f}\colon \boldsymbol{F}(X) \to G$, and if $X \precsim Y$ then $\boldsymbol{F}(X)$ can be identified with a subgroup of $\boldsymbol{F}(Y)$.

If X is well-orderable, then the **rank** of $\boldsymbol{F}(X)$ is the cardinality of X. If $X \asymp Y$ then $\boldsymbol{F}(X) \cong \boldsymbol{F}(Y)$, and the unique (up to isomorphism) free group of rank $\kappa \neq 0$ is $\boldsymbol{F}_\kappa$. (If $X \precsim Y$ then $\boldsymbol{F}(X)$ is isomorphic to a subgroup of $\boldsymbol{F}(Y)$, but the conversely.) Summarizing:

Proposition 18.38

Assume AC. If X, Y are infinite sets, then
$|X| = |Y| \Leftrightarrow \boldsymbol{F}(X) \cong \boldsymbol{F}(Y) \Leftrightarrow |\boldsymbol{F}(X)| = |\boldsymbol{F}(Y)|$.