Elements of Mathematical Logic

Section 19 of Chapter V

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Theorem 19.1

Let A be a *class*, let $\bar{a} \in A$ and let $F \colon \omega \times A \to A$ be a functional relation. There is a unique $G \colon \omega \to A$ such that

$$\begin{cases} G(0) = \bar{a} \\ G(\mathbf{S}(n)) = F(n, G(n)). \end{cases}$$

Proof

Let $\mathcal{G} = \{p \mid \exists m \in \omega \big[p \colon m \to A \wedge (0 < m \Rightarrow p(0) = \bar{a}) \wedge \forall n\, (\mathbf{S}(n) < m \Rightarrow p(\mathbf{S}(n)) = F(n, p(n)))\big]\}$.

Claim

If $p, q \in \mathcal{G}$ then $p \cup q$ is a function.

Proof.

Suppose that $p, q \in \mathcal{G}$ but $p \cup q$ is not a function. Then there is a least $n \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$ witnessing that $p(n) \neq q(n)$. Clearly $n \neq 0$, since $p(0) = \bar{a} = q(0)$, so $n = \mathbf{S}(k)$. Then

$$\begin{aligned} p(n) &= F(k, p(k)) \\ &= F(k, q(k)) && \text{by minimality of } n, \\ &= q(n). \end{aligned}$$

□

Proof

Thus $G = \bigcup \mathcal{G} \subseteq \omega \times A$ is a functional relation, and therefore a function by replacement.
As $\{(0, \bar{a})\} \in \mathcal{G}$, it follows that $G \neq \emptyset$ and $G(0) = \bar{a}$. Moreover if $\mathbf{S}(n) \in \operatorname{dom}(G)$ then $G(\mathbf{S}(n)) = p(\mathbf{S}(n))$ for some $p \in \mathcal{G}$, so $G(\mathbf{S}(n)) = F(n, p(n)) = F(n, G(n))$.

$\operatorname{dom}(G) = \omega$

Towards a contradiction, suppose $\bar{n}$ is least such that $\bar{n} \notin \operatorname{dom}(G)$, then $\bar{n} = \mathbf{S}(\bar{m})$ for some $\bar{m}$. As $\bar{n} \notin \operatorname{dom}(p)$ for any $p \in \mathcal{G}$, it follows that $k \notin \operatorname{dom}(G)$ for all $k > \bar{n}$. It is easy to check that $p \stackrel{\text{def}}{=} G \cup \{(\bar{n}, F(\bar{m}, G(\bar{m})))\} \in \mathcal{G}$, thus $p \subseteq G$, so that $\bar{n} \in \operatorname{dom}(G)$: a contradiction.

Proof

G is unique

If G' were another function satisfying the statement of theorem, then let $\bar{n}$ be least such that $G(\bar{n}) \neq G'(\bar{n})$. Clearly $\bar{n} \neq 0$ so $\bar{n} = \mathbf{S}(\bar{m})$ for some $\bar{m}$, and hence

$$\begin{aligned} G(\bar{n}) &= F(\bar{m}, G(\bar{m})) \\ &= F(\bar{m}, G'(\bar{m})) && \text{by minimality of } \bar{n}, \\ &= G'(\bar{n}), \end{aligned}$$

a contradiction!

Transitive closure of a relation

The **transitive closure** of $R \subseteq X \times X$ with X a class, is the relation

$$\tilde{R} = \Big\{(x, y) \in X \times X \mid \exists n > 0 \, \exists f \in {}^{\mathbf{S}(n)}X \big[x = f(0) \wedge \\ y = f(n) \wedge \forall i < n \, (f(i), f(\mathbf{S}(i))) \in R\big]\Big\}$$

In other words $x \mathrel{\tilde{R}} y$ if and only there are $x_0, \ldots, x_n$ such that

$$x = x_0 \mathrel{R} x_1 \cdots x_{n-1} \mathrel{R} x_n = y.$$

Proposition 19.2

R is regular on X if and only if $\tilde{R}$ is regular on X.

Proof.

Since $R \subseteq \tilde{R}$ it is enough to check that if R is regular, then $\tilde{R}$ is regular. Given a $\bar{x} \in X$, by induction define the sets Z_n

$$Z_0 = \{y \in X \mid y \, R \, \bar{x}\}$$
$$Z_{n+1} = \{y \in X \mid \exists z \in Z_n \, (y \, R \, z)\} = \bigcup_{z \in Z_n} \{y \in X \mid y \, R \, z\}.$$

The sequence $\langle Z_n \mid n \in \omega \rangle$ is given by Theorem 19.1 when $A = \mathrm{V}$, $\bar{a} = Z_0$ and $F(n, a) = F(a) = \{x \in X \mid \exists y \in a \, (x \, R \, y)\}$. Thus $G(n) = Z_n$. By replacement $\bigcup_{n \in \omega} Z_n$ is a set, and it is the same as $\{y \in X \mid y \, \tilde{R} \, \bar{x}\}$. □

Proposition 19.3

R is well-founded on X if and only if $\tilde{R}$ is well-founded on X.

Proof.

Since $R \subseteq \tilde{R}$ the $\Leftarrow$ direction is clear.
Suppose R is well-founded towards proving that so is $\tilde{R}$. Fix $\emptyset \neq Y \subseteq X$ and let us show that there is an $\tilde{R}$-minimal element in Y. A path from Y into itself is a sequence $\langle z_0, \ldots, z_n, z_{n+1} \rangle$ in X of length ≥ 1 such that $z_0, z_{n+1} \in Y$ and $z_i \; R \; z_{i+1}$ with $i = 0, \ldots, n$. (Paths of length 1 can be identified with the elements of Y.) Let
$\bar{Y} = \{x \in X \mid \exists s \, (s \text{ is a path from } Y \text{ into itself and } x \in \operatorname{ran} s)\}$. By construction $Y \subseteq \bar{Y}$ and let $\bar{y}$ be an R-minimal of $\bar{Y}$. By construction no element of $\bar{Y} \setminus Y$ is R-minimal, so $\bar{y} \in Y$. Let us check that $\bar{y}$ is $\tilde{R}$-minimal in Y. Towards contradiction, if $\bar{x} \; \tilde{R} \; \bar{y}$ for some $\bar{x} \in Y$ different from $\bar{y}$, then there would be a path $\langle z_0, \ldots, z_{n+1} \rangle$ from Y into itself with $z_0 = \bar{x}$ and $z_{n+1} = \bar{y}$, and hence $z_n \; R \; \bar{y}$, against R-minimality of $\bar{y}$. □

Theorem 19.4

Let X and Z be classes, let $R \subseteq X \times X$ be irreflexive, regular, and well-founded, and let $F\colon Z \times X \times \mathrm{V} \to \mathrm{V}$. Then there is a unique $G\colon Z \times X \to \mathrm{V}$ such that for every $(z, x) \in Z \times X$

$$G(z, x) = F(z, x, G \restriction \{(z, y) \mid y \mathrel{R} x\}). \qquad (*)$$

Proof of uniqueness

Suppose that $G, G'\colon Z \times X \to \mathrm{V}$ satisfy $(*)$ and that $G \neq G'$. Fix $\bar{z} \in Z$ such that $Y = \{x \in X \mid G(\bar{z}, x) \neq G'(\bar{z}, x)\} \neq \emptyset$ and let $\bar{x} \in Y$ be an R-minimal element of Y. Then

$$G \restriction \{(\bar{z}, y) \mid y \mathrel{R} \bar{x}\} = G' \restriction \{(\bar{z}, y) \mid y \mathrel{R} \bar{x}\}$$

and let $\bar{p}$ be this functional relation. Regularity of R together with the axiom of replacement, imply that $\bar{p}$ is a set, and therefore $G(\bar{z}, \bar{x}) = F(\bar{z}, \bar{x}, \bar{p}) = G'(\bar{z}, \bar{x})$: a contradiction.

Proof of existence

Let $\mathcal{G}$ be the class of all functions p such that

1. $\mathrm{dom}(p) \subseteq Z \times X$,
2. $\forall (z, x) \in \mathrm{dom}(p) \, \forall y \in X \, (y \mathrel{R} x \Rightarrow (z, y) \in \mathrm{dom}(p))$,
3. $\forall (z, x) \in \mathrm{dom}(p) \, (p(z, x) = F(z, x, p \restriction \{(z, y) \mid y \mathrel{R} x\}))$.

Proof of existence

Let $\mathcal{G}$ be the class of all functions p such that

1. $\operatorname{dom}(p) \subseteq Z \times X$,
2. $\forall (z, x) \in \operatorname{dom}(p) \, \forall y \in X \, (y \mathrel{R} x \Rightarrow (z, y) \in \operatorname{dom}(p))$,
3. $\forall (z, x) \in \operatorname{dom}(p) \, (p(z, x) = F(z, x, p \restriction \{(z, y) \mid y \mathrel{R} x\}))$.

2 is equivalent to the seemingly stronger condition
$\forall (z, x) \in \operatorname{dom}(p) \, (\{z\} \times \{y \in X \mid y \mathrel{\tilde{R}} x\} \subseteq \operatorname{dom}(p))$,
where $\tilde{R}$ is the transitive closure of R.

Proof of existence

Let $\mathcal{G}$ be the class of all functions p such that

1. $\mathrm{dom}(p) \subseteq Z \times X$,
2. $\forall (z,x) \in \mathrm{dom}(p)\, \forall y \in X\, (y\ R\ x \Rightarrow (z,y) \in \mathrm{dom}(p))$,
3. $\forall (z,x) \in \mathrm{dom}(p)\, (p(z,x) = F(z,x,p \restriction \{(z,y) \mid y\ R\ x\}))$.

2 is equivalent to the seemingly stronger condition
$\forall (z,x) \in \mathrm{dom}(p)\, (\{z\} \times \{y \in X \mid y\ \tilde{R}\ x\} \subseteq \mathrm{dom}(p))$,
where $\tilde{R}$ is the transitive closure of R.

If p satisfies 2 , $(z,x) \in \mathrm{dom}(p)$ and
$y\ \tilde{R}\ xy = w_0\ R\ w_1\ R \ldots R\ w_n\ R\ w_{n+1} = x$, then by induction
$(z, w_i) \in \mathrm{dom}(p)$ for all $i \leq n$.

Proof of existence

Let $\mathcal{G}$ be the class of all functions p such that

1. $\mathrm{dom}(p) \subseteq Z \times X$,
2. $\forall (z, x) \in \mathrm{dom}(p) \, \forall y \in X \, (y \; R \; x \Rightarrow (z, y) \in \mathrm{dom}(p))$,
3. $\forall (z, x) \in \mathrm{dom}(p) \, (p(z, x) = F(z, x, p \restriction \{(z, y) \mid y \; R \; x\}))$.

2 is equivalent to the seemingly stronger condition
$\forall (z, x) \in \mathrm{dom}(p) \, (\{z\} \times \{y \in X \mid y \; \tilde{R} \; x\} \subseteq \mathrm{dom}(p))$,
where $\tilde{R}$ is the transitive closure of R.

If $p, q \in \mathcal{G}$ then $p \cup q$ is a function

If $\{x \in X \mid \exists z \in Z \, ((z, x) \in \mathrm{dom}(p) \cap \mathrm{dom}(q) \; \wedge \; p(z, x) \neq q(z, x))\} \neq \emptyset$ by well-foundedness let $\bar{x}$ be an R-minimal element of this class. Let $\bar{z} \in Z$ be such that $(\bar{z}, \bar{x}) \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and $p(\bar{z}, \bar{x}) \neq q(\bar{z}, \bar{x})$. By 2 $\{(\bar{z}, y) \mid y \; R \; \bar{x}\} \subseteq \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and by R-minimality of $\bar{x}$ $p \restriction \{(\bar{z}, y) \mid y \; R \; \bar{x}\} = q \restriction \{(\bar{z}, y) \mid y \; R \; \bar{x}\} \stackrel{\text{def}}{=} \bar{r}$ so that, by 3 $p(\bar{z}, \bar{x}) = F(\bar{z}, \bar{x}, \bar{r}) = q(\bar{z}, \bar{x})$: a contradiction.

Proof of existence

If $p, q \in \mathcal{G}$ then $p \cup q \in \mathcal{G}$, and hence $G = \bigcup \mathcal{G}$ is a functional relation with domain $\subseteq Z \times X$. If $Z \times X \setminus \mathrm{dom}(G) \neq \emptyset$, let $\bar{x}$ be an R-minimal element of $\{x \in X \mid \exists z \in Z \ (z, x) \notin \mathrm{dom}(G)\}$ and let $\bar{z} \in Z$ be such that $(\bar{z}, \bar{x}) \notin \mathrm{dom}(G)$. The transitive closure $\tilde{R}$ of R on X, is regular, so

$$\bar{p} \stackrel{\text{def}}{=} G \restriction \{(\bar{z}, y) \mid y \; \tilde{R} \; \bar{x}\}$$

is a set by the axiom of replacement. Then $\bar{p} \in \mathcal{G}$ and hence $\bar{p} \cup \{((\bar{z}, \bar{x}), F(\bar{z}, \bar{x}, \bar{p}))\} \in \mathcal{G}$. Thus $(\bar{z}, \bar{x}) \in \mathrm{dom}(G)$, against our assumption. It follows that G is the functional relation we are looking for.

Rank of a well-founded relation

If R is irreflexive, regular, and well-founded on X, then $\boldsymbol{\varrho}_{R,X} \colon X \to \mathrm{Ord}$ defined by $\boldsymbol{\varrho}_{R,X}(x) = \bigcup \{ \mathbf{S}(\boldsymbol{\varrho}_{R,X}(y)) \mid y \mathrel{R} x \}$ is the **rank of** R **on** X.

Proposition 19.6

$\mathrm{ran}(\boldsymbol{\varrho}_{R,X})$ is an initial segment of Ord and

1. $x \mathrel{R} y \Rightarrow \boldsymbol{\varrho}_{R,X}(x) < \boldsymbol{\varrho}_{R,X}(y)$,
2. $\boldsymbol{\varrho}_{R,X}(x) = \inf \{ \alpha \mid \forall y \, \big(y \mathrel{R} x \Rightarrow \boldsymbol{\varrho}_{R,X}(y) < \alpha \big) \}$.

Proof.

If $\boldsymbol{\varrho}_{R,X}(y) \in \mathrm{Ord}$ for any y such that $y \mathrel{R} x$, then $\boldsymbol{\varrho}_{R,X}(x) \in \mathrm{Ord}$ by Proposition 18.6, so $\mathrm{ran}(\boldsymbol{\varrho}_{R,X}) \subseteq \mathrm{Ord}$. Towards a contradiction, suppose there is $\bar{x} \in X$ and α such that $\alpha \in \boldsymbol{\varrho}_{R,X}(x) \setminus \mathrm{ran}(\boldsymbol{\varrho}_{R,X})$, and let $\bar{x}$ be R-minimal such. Then there is $y \mathrel{R} \bar{x}$ such that $\alpha < \mathbf{S}(\boldsymbol{\varrho}_{R,X}(y))$. Since $\alpha \notin \mathrm{ran}\, \boldsymbol{\varrho}_{R,X}$ then $\alpha < \boldsymbol{\varrho}_{R,X}(y)$, against R-minimality of $\bar{x}$. The rest of the proof is easy. □

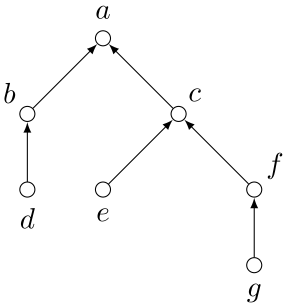
If R is an irreflexive, regular, well-founded relation on X, the function $\boldsymbol{\pi}_{R,X} \colon X \to \mathrm{V}$ given by

$$\boldsymbol{\pi}_{R,X}(x) = \{\boldsymbol{\pi}_{R,X}(y) \mid y \mathrel{R} x\}$$

is the **Mostowski collapsing function**.
The class $\overline{X} = \operatorname{ran}(\boldsymbol{\pi}_{R,X})$ is the **Mostowski collapse of R and X**.

1. $\overline{X}$ is transitive and
2. $\forall x, y \in X \, (x \mathrel{R} y \Rightarrow \boldsymbol{\pi}_{R,X}(x) \in \boldsymbol{\pi}_{R,X}(y))$.

If R is the relation on $\{a, b, c, d, e, f, g\}$ given by the directed graph



then $\boldsymbol{\varrho}_{R,X}(d) = \boldsymbol{\varrho}_{R,X}(e) = \boldsymbol{\varrho}_{R,X}(g) = 0$, $\boldsymbol{\varrho}_{R,X}(b) = \boldsymbol{\varrho}_{R,X}(f) = 1$, $\boldsymbol{\varrho}_{R,X}(c) = 2$ and $\boldsymbol{\varrho}_{R,X}(a) = 3$;
$\boldsymbol{\pi}_{R,X}(d) = \boldsymbol{\pi}_{R,X}(e) = \boldsymbol{\pi}_{R,X}(g) = \emptyset$, $\boldsymbol{\pi}_{R,X}(b) = \boldsymbol{\pi}_{R,X}(f) = \{\emptyset\} = 1$, $\boldsymbol{\pi}_{R,X}(c) = \{0, 1\} = 2$ and $\boldsymbol{\pi}_{R,X}(a) = \{1, 2\}$.

Definition 19.7

$R \subseteq X \times X$ is **extensional on** X if

$$\forall x, y \in X\, (\forall z \in X\, (z \mathrel{R} x \Leftrightarrow z \mathrel{R} y) \Rightarrow x = y)\,.$$

Example

1. If X is a transitive class, then $\in \restriction X = \{(y, x) \in X \times X \mid y \in x\}$ is extensional on X;
2. if R is a (strict) linear order on X, then R is extensional on X.

Proposition 19.8

Let R be an irreflexive, regular, well-founded relation on the class X.

1. If R is extensional on X, then $\boldsymbol{\pi}_{R,X}$ is injective and $\boldsymbol{\pi}_{R,X} \colon X \to \overline{X}$ is an isomorphism between R on X and $\in$ on $\overline{X}$, that is $\boldsymbol{\pi}_{R,X}$ is bijective and $\forall x, y \in X\, (x \mathrel{R} y \Leftrightarrow \boldsymbol{\pi}_{R,X}(x) \in \boldsymbol{\pi}_{R,X}(y))$.
2. If R is a strict well-order on X the functions $\boldsymbol{\pi}_{R,X}$ and $\boldsymbol{\varrho}_{R,X}$ agree.

Proposition 19.8 [1]

Let R be extensional, irreflexive, regular, and well-founded on the class X. Then $\boldsymbol{\pi}_{R,X} \colon X \to \overline{X}$ is bijective and $\forall x, y \in X (x \mathrel{R} y \Leftrightarrow \boldsymbol{\pi}_{R,X}(x) \in \boldsymbol{\pi}_{R,X}(y))$.

Proof.

Towards a contradiction, let $\bar{x}$ be R-minimal such that $\boldsymbol{\pi}_{R,X}(\bar{x}) = \boldsymbol{\pi}_{R,X}(\bar{y})$, for some $\bar{y} \neq \bar{x}$. Let $z \mathrel{R} \bar{x}$: as $\boldsymbol{\pi}_{R,X}(z) \in \boldsymbol{\pi}_{R,X}(\bar{x}) = \boldsymbol{\pi}_{R,X}(\bar{y})$, there is $w \mathrel{R} \bar{y}$ such that $\boldsymbol{\pi}_{R,X}(z) = \boldsymbol{\pi}_{R,X}(w)$. By minimality of $\bar{x}$, $z = w$. Thus $z \mathrel{R} \bar{x} \Rightarrow z \mathrel{R} \bar{y}$. Similarly, if $z \mathrel{R} \bar{y}$ then there is $w \mathrel{R} \bar{x}$ such that $\boldsymbol{\pi}_{R,X}(z) = \boldsymbol{\pi}_{R,X}(w)$ and hence $z = w$, that is $z \mathrel{R} \bar{y} \Rightarrow z \mathrel{R} \bar{x}$. Thus, by extensionality, $\bar{y} = \bar{x}$, against our assumption. It follows that $\boldsymbol{\pi}_{R,X}$ is a bijection between X and $\overline{X}$.

If $\boldsymbol{\pi}_{R,X}(x) \in \boldsymbol{\pi}_{R,X}(y) = \{\boldsymbol{\pi}_{R,X}(z) \mid z \mathrel{R} y\}$, then $x \mathrel{R} y$. □

Proposition 19.8 ❷

Let R be irreflexive, regular and well-founded on X. If R is a strict well-order on X, then the functional relations $\boldsymbol{\pi}_{R,X}$ and $\boldsymbol{\varrho}_{R,X}$ coincide.

Proof.

Assume $\boldsymbol{\varrho}_{R,X}(y) = \boldsymbol{\pi}_{R,X}(y)$, for all $y \mathrel{R} x$. Then $\boldsymbol{\pi}_{R,X}(x) = \{\boldsymbol{\pi}_{R,X}(y) \mid y \mathrel{R} x\} = \{\boldsymbol{\varrho}_{R,X}(y) \mid y \mathrel{R} x\}$ is a set of ordinals. If $\boldsymbol{\pi}_{R,X}(z) \in \boldsymbol{\pi}_{R,X}(y) \in \boldsymbol{\pi}_{R,X}(x)$, then $z \mathrel{R} y \mathrel{R} x$, so $z \mathrel{R} x$, that is $\boldsymbol{\pi}_{R,X}(x)$ is transitive, and therefore it is an ordinal. By construction $\boldsymbol{\pi}_{R,X}(x)$ is the least upper bound of the ordinals $\mathbf{S}(\boldsymbol{\pi}_{R,X}(y)) = \mathbf{S}(\boldsymbol{\varrho}_{R,X}(y))$ with $y \mathrel{R} x$, that is $\boldsymbol{\pi}_{R,X}(x) = \boldsymbol{\varrho}_{R,X}(x)$. □

Lemma 19.9

If $f\colon \mathrm{Ord} \to \mathrm{Ord}$ is increasing and continuous, then

$$\forall \alpha \exists \bar{\alpha} > \alpha \, (f(\bar{\alpha}) = \bar{\alpha}) \, .$$

Proof.

By recursion define $\langle \alpha_n \mid n \in \omega \rangle$ by $\alpha_0 = \mathbf{S}(\alpha)$ and $\alpha_{\mathbf{S}(n)} = f(\alpha_n)$, and let $\bar{\alpha} = \sup_n \alpha_n$. If $f(\alpha_0) = \alpha_0$, then $\forall n \, (\alpha_0 = \alpha_n)$ and therefore $\bar{\alpha} = \alpha_0$. If instead $\alpha_0 < f(\alpha_0) = \alpha_1$, then $\alpha_n < \alpha_{\mathbf{S}(n)}$, and hence $\bar{\alpha}$ is limit. Then

$$f(\bar{\alpha}) = \sup_{\nu < \bar{\alpha}} f(\nu) = \sup_n f(\alpha_n) = \sup_n \alpha_{\mathbf{S}(n)} = \bar{\alpha}.$$

In either case $\bar{\alpha}$ is the least fixed point greater that α for f. □

Definition

$\aleph \colon \mathrm{Ord} \to \mathrm{Card} \setminus \omega$ is the class-function enumerating the class of infinite cardinals, i.e.

$$\begin{aligned} \aleph_0 &= \omega \\ \aleph_{\mathbf{S}(\alpha)} &= (\aleph_\alpha)^+ \\ \aleph_\lambda &= \sup_{\alpha<\lambda} \aleph_\alpha . \end{aligned}$$

Since $\aleph \colon \mathrm{Ord} \to \mathrm{Ord}$ is increasing and continuous, there are cardinals κ such that $\kappa = \aleph_\kappa$, and the least such is the least upper bound of

$$\aleph_0, \quad \aleph_{\aleph_0}, \quad \aleph_{\aleph_{\aleph_0}}, \quad \aleph_{\aleph_{\aleph_{\aleph_0}}}, \quad \dots$$

Ordinal arithmetic

$$\alpha \dot{+} \beta = \begin{cases} \alpha & \text{if } \beta = 0, \\ \mathbf{S}(\alpha \dot{+} \gamma) & \text{if } \beta = \mathbf{S}(\gamma), \\ \sup_{\gamma < \beta}(\alpha \dot{+} \gamma) & \text{if } \beta \text{ is limit,} \end{cases}$$

$$\alpha \cdot \beta = \begin{cases} 0 & \text{if } \beta = 0, \\ (\alpha \cdot \gamma) \dot{+} \alpha & \text{if } \beta = \mathbf{S}(\gamma), \\ \sup_{\gamma < \beta} \alpha \cdot \gamma & \text{if } \beta \text{ is limit,} \end{cases}$$

$$\alpha^{\cdot \beta} = \begin{cases} 1 & \text{if } \beta = 0, \\ \alpha^{\cdot \gamma} \cdot \alpha & \text{if } \beta = \mathbf{S}(\gamma), \\ \sup_{\gamma < \beta} \alpha^{\cdot \gamma} & \text{if } \beta \text{ is limit.} \end{cases}$$

The ordinal $\varrho_{R,X}(x)$, when $X = \mathrm{V}$ and R is the membership relation, is called the **rank of** x, denoted by $\mathrm{rank}(x)$. By Proposition 19.6

$$x \in y \Rightarrow \mathrm{rank}(x) < \mathrm{rank}(y) \qquad x \subseteq y \Rightarrow \mathrm{rank}(x) \leq \mathrm{rank}(y)$$

and by induction one checks that $\mathrm{rank}(\alpha) = \alpha$.

Proposition 19.10

1. $\mathrm{rank}(\mathscr{P}(x)) = \mathbf{S}(\mathrm{rank}(x))$.
2. $\mathrm{rank}(\bigcup x) = \sup\{\mathrm{rank}(y) \mid y \in x\}$.

Proof.

1. Since $x \in \mathscr{P}(x)$ one has that $\mathbf{S}(\mathrm{rank}(x)) \leq \mathrm{rank}(\mathscr{P}(x))$. Conversely, if $y \subseteq x$, then $\mathbf{S}(\mathrm{rank}(y)) \leq \mathbf{S}(\mathrm{rank}(x))$ and hence $\mathrm{rank}(\mathscr{P}(x)) = \sup\{\mathbf{S}(\mathrm{rank}(y)) \mid y \subseteq x\} \leq \mathbf{S}(\mathrm{rank}(x))$. □

Proposition 19.10

1. $\mathrm{rank}(\mathscr{P}(x)) = \mathbf{S}(\mathrm{rank}(x))$.
2. $\mathrm{rank}(\bigcup x) = \sup\{\mathrm{rank}(y) \mid y \in x\}$.

Proof.

2. If $y \in x$ then $y \subseteq \bigcup x$ so $\sup\{\mathrm{rank}(y) \mid y \in x\} \leq \mathrm{rank}(\bigcup x)$.

Conversely, if $z \in y \in x$ then $\mathbf{S}(\mathrm{rank}(z)) \leq \mathrm{rank}(y)$ so $\mathbf{S}(\mathrm{rank}(z)) \leq \sup\{\mathrm{rank}(y) \mid y \in x\}$. Being z arbitrary, $\mathrm{rank}(\bigcup x) \leq \sup\{\mathrm{rank}(y) \mid y \in x\}$. □

Definition 19.11

$\mathrm{V}_\alpha = \{x \mid \mathrm{rank}(x) < \alpha\}$.

Theorem 19.12

V_α is a transitive set and

$$\mathrm{V}_\alpha = \bigcup_{\beta<\alpha} \mathscr{P}(\mathrm{V}_\beta). \quad (*)$$

Proof.

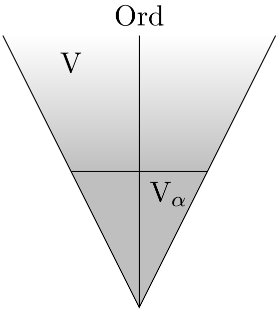
If $y \in x \in \mathrm{V}_\alpha$ then $\mathrm{rank}(y) < \mathrm{rank}(x) < \alpha$ so $y \in \mathrm{V}_\alpha$. Thus V_α is a transitive class. By induction on α we show that V_α is a set and that (*) holds. Suppose the results holds true for $\beta < \alpha$: then $\{\mathscr{P}(\mathrm{V}_\beta) \mid \beta < \alpha\}$ is a set, so it is enough to show (*).

$x \subseteq \mathrm{V}_{\mathrm{rank}(x)}$ and therefore $\mathrm{rank}(x) < \alpha \Rightarrow x \in \bigcup_{\beta<\alpha} \mathscr{P}(\mathrm{V}_\beta)$.

Conversely, if $x \in \bigcup_{\beta<\alpha} \mathscr{P}(\mathrm{V}_\beta)$, then $x \subseteq \mathrm{V}_\beta$, for some $\beta < \alpha$ and hence $\mathrm{rank}(y) < \beta$ for any $y \in x$, whence $\mathrm{rank}(x) \leq \beta < \alpha$. □

Corollary 19.13

1. $\mathrm{V}_0 = \emptyset$.
2. If $\alpha < \beta$ then $\mathrm{V}_\alpha \in \mathrm{V}_\beta$ and $\mathrm{V}_\alpha \subset \mathrm{V}_\beta$.
3. $\mathrm{V}_{\mathbf{S}(\alpha)} = \mathscr{P}(\mathrm{V}_\alpha)$.
4. $\mathrm{V}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{V}_\alpha$, if λ is limit.
5. $\mathrm{V} = \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{V}_\alpha$.



Models of set theory

A structure for $\mathcal{L}_{\in}$ is a pair $\langle M, E\rangle$ where M is a nonempty set and $E \subseteq M \times M$. In particular, consider the structure V_α, i.e. $\langle \mathrm{V}_\alpha, \in\rangle$, with $\alpha > 0$.

Question

Which axioms of ZF are true in V_α?

Theorem 19.15

1. All axioms of ZFC except the axiom of infinity hold in V_ω.
2. All axioms of ZF except possibly for replacement hold in V_λ, if $\lambda > \omega$ is limit.
3. Assuming choice, then AC holds in V_λ, if λ is limit.

In order to prove this result a stratification of formulæ is introduced.

Definition 19.17

An $\mathcal{L}_\in$-formula is Δ_0 if it belongs to the smallest class containing all atomic formulæ and closed under connectives and **bounded quantifications**, that is

- atomic formulæ are Δ_0,
- if φ, ψ are Δ_0 then so are $\neg\varphi$ and $\varphi \odot \psi$, where $\odot$ is any binary connective,
- if φ is Δ_0 then so is $\forall y(y \in x \Rightarrow \varphi)$ and $\exists y(y \in x \wedge \varphi)$,

and nothing else is a Δ_0-formula.

The axiom of extensionality is true in every transitive set $M \neq \emptyset$.

We write $\forall y \in x\, \varphi$ and $\exists y \in x\, \varphi$ instead of $\forall y(y \in x \Rightarrow \varphi)$ and $\exists y(y \in x \wedge \varphi)$.

Examples of Δ_0 formulæ

Trans(x), i.e. x is transitive	$x = \{y, z\}$
Ord(x), i.e. x is an ordinal	$x = (y, z)$
Op(x), i.e x is an ordered pair	$f \colon x \to y$
Rel(x), i.e. x is a relation	$y = \mathrm{dom}(x)$
Fn(x), i.e. x is a function	$y = \mathrm{ran}(x)$
Seq(x), i.e. x is a finite sequence	$\mathbf{S}(x) = y$
x is an injective function	$f(x) = g(y)$
x is a reflexive relation	$g = f \restriction x$
x is a symmetric relation	$f(x) = y$
x is a transitive relation	$f``x = y$
$x \subseteq y$	$z = x \times y$
$z = x \cup y$	$z = x \setminus y$
$z = x \cap y$	

Definition 19.18

A $\mathcal{L}_\in$-formula is Σ_1 if it is of the form $\exists x\, \varphi$ with φ a Δ_0-formula; it is Π_1 if it is of the form $\forall x\, \varphi$ with φ a Δ_0-formula.

Definition 19.19

Let M be a non-empty set. We say that $\varphi(x_1, \dots, x_n)$ is:

- **upward absolute between** M **and** V if
  $\forall a_1, \dots, a_n \in M\, \big((\langle M, \in\rangle \vDash \varphi[a_1, \dots, a_n]) \Rightarrow \varphi(a_1, \dots, a_n)\big)$;
- **downward absolute between** M **and** V if
  $\forall a_1, \dots, a_n \in M\, \big(\varphi(a_1, \dots, a_n) \Rightarrow (\langle M, \in\rangle \vDash \varphi[a_1, \dots, a_n])\big)$;
- **absolute between** M **and** V if it is both upward and downward absolute, that is
  $\forall a_1, \dots, a_n \in M\, \big((\langle M, \in\rangle \vDash \varphi[a_1, \dots, a_n]) \Leftrightarrow \varphi(a_1, \dots, a_n)\big)$,

where $\varphi(a_1, \dots, a_n)$ stands for $\varphi(\!|a_1/x_1, \dots, a_n/x_n|\!)$.

From the definition it follows that φ is upward absolute between M and V if and only if $\neg\varphi$ is downward absolute between M and V, and that if φ and ψ are upward/downward absolute, then so are $\varphi \wedge \psi$ and $\varphi \vee \psi$. Therefore the collection of formulæ that are absolute between M and V is closed under all connectives.
An easy induction on the complexity of formulæ yields

Lemma 19.20

A quantifier-free formula is absolute between transitive $M \neq \emptyset$ and V.

Lemma 19.21

Suppose M is a non-empty transitive set.

1. Every Δ_0 formula is absolute between M and V.
2. Every Σ_1 formula is upward absolute between M and V, and every Π_1 formula is downward absolute between M and V.

Every Δ_0 formula is absolute between M and V.

By Lemma 19.20 it is enough to consider formulæ of the form $\forall y \in x_i\, \varphi(y, x_1, \ldots, x_n)$. Fix $a_1, \ldots, a_n \in M$. By the inductive hypothesis, and since M is transitive,

$$\begin{aligned} \langle M, \in \rangle \vDash \forall y \in x_i\, \varphi[\vec{a}] &\Leftrightarrow \forall b \in M\, (b \in a_i \Rightarrow \langle M, \in \rangle \vDash \varphi[b, \vec{a}]) \\ &\Leftrightarrow \forall b \in a_i\, \langle M, \in \rangle \vDash \varphi[b, \vec{a}] \\ &\Leftrightarrow \forall y \in a_i\, \varphi(\vec{a}). \end{aligned}$$

Every Σ_1 formula is upward absolute between M and V, and every Π_1 formula is downward absolute between M and V.

It is enough to prove that Σ_1 formulæ are upward absolute. Suppose that $\varphi(y_1, \ldots, y_k, x_1, \ldots, x_n)$ is Δ_0, that $a_1, \ldots, a_n \in M$, and that $\langle M, \in \rangle \vDash \exists y_1, \ldots, y_k\, \varphi[a_1, \ldots, a_n]$. Fix $b_1, \ldots, b_k \in M$ such that $\langle M, \in \rangle \vDash \varphi[b_1, \ldots, b_k, a_1, \ldots, a_n]$. By the preceding point $\varphi(b_1, \ldots, b_k, a_1, \ldots, a_n)$ holds, and hence $\exists y_1, \ldots, y_k\, \varphi(a_1, \ldots, a_n)$.

Theorem 19.22

Suppose $M \neq \emptyset$ is a transitive set. Then

1. $\langle M, \in \rangle$ satisfies the axioms of extensionality and foundation.
2. If $\{a, b\} \in M$ for all $a, b \in M$, then $\langle M, \in \rangle$ satisfies the axiom of pairing.
3. If $\bigcup a \in M$ for all $a \in M$, then $\langle M, \in \rangle$ satisfies the axiom of union.
4. If $\forall a \in M \, (\mathscr{P}(a) \cap M \in M)$, then $\langle M, \in \rangle$ satisfies the power-set axiom.
5. If $\omega \in M$ then $\langle M, \in \rangle$ satisfies the axiom of infinity.
6. If $\forall a \in M \, \forall b \subseteq a \, (b \in M)$, then $\langle M, \in \rangle$ satisfies the axiom schema of separation.
7. If for all $a \in M$ and all $f \colon a \to M$ there is $b \in M$ such that $\operatorname{ran} f \subseteq b$, then $\langle M, \in \rangle$ satisfies the axiom schema of replacement.
8. $\langle M, \in \rangle \vDash$ AC if and only if $\forall \mathcal{A} \in M \, (\forall A \in \mathcal{A} \, (A \neq \emptyset) \Rightarrow \exists f \in M \, (f$ is a choice function for $\mathcal{A}))$.

$\langle M, \in\rangle$ satisfies the axioms of extensionality and foundation.

The axioms of extensionality and foundations are the universal closure of the Δ_0-formulæ

$$\forall z \in x\,(z \in y) \wedge \forall z \in y\,(z \in x) \Rightarrow x = y$$
$$\exists y \in x\,(y = y) \Rightarrow \exists y \in x\,\forall z \in y\,(z \notin x)$$

so they are downward absolute. Both axioms hold in V and therefore hold in $\langle M, \in\rangle$.

If $\{a, b\} \in M$ for all $a, b \in M$, then $\langle M, \in\rangle$ satisfies the axiom of pairing.

$z = \{x, y\}$ is Δ_0.

If $\bigcup a \in M$ for all $a \in M$, then $\langle M, \in\rangle$ satisfies the axiom of union.

$v = \bigcup u$ is Δ_0.

If $\forall a \in M\ (\mathscr{P}(a) \cap M \in M)$, then $\langle M, \in \rangle$ satisfies the power-set axiom.

Fix $a \in M$ and let $b \stackrel{\text{def}}{=} \mathscr{P}(a) \cap M$. As $z \subseteq x$ is Δ_0, then $\langle M, \in \rangle$ satifies $\forall z (z \subseteq x \Leftrightarrow z \in y)$, where x and y are given the values a and b

If $\omega \in M$ then $\langle M, \in \rangle$ satisfies the axiom of infinity.

The axiom of infinity is $\exists x\, \varphi(x)$ where $\varphi(x)$ is the Δ_0-formula $\emptyset \in x \wedge \forall y \in x\, (\mathbf{S}(y) \in x)$, so by absoluteness $\langle M, \in \rangle$ satisfies the axiom of infinity if and only if $\exists x \in M\ \varphi(x)$. As ω satisfies φ, if $\omega \in M$ then $\langle M, \in \rangle$ satisfies the axiom of infinity.

If $\forall a \in M\, \forall b \subseteq a\, (b \in M)$, then $\langle M, \in \rangle$ satisfies the axiom schema of separation.

We must show that given $\varphi(x, y, \vec{w})$, and given $a, \vec{c} \in M$ to be assigned to the variables $y, \vec{w}$, the set $b = \{d \in a \mid \langle M, \in \rangle \vDash \varphi[d, a, \vec{c}]\}$ belongs to M. But this follows at once by the assumption and by $b \subseteq a$.

If for all $a \in M$ and all $f\colon a \to M$ there is $b \in M$ such that $\operatorname{ran} f \subseteq b$, then $\langle M, \in \rangle$ satisfies the axiom schema of replacement.

We must show that given $\varphi(x, y, z, \vec{w})$ and given $a, \vec{c} \in M$ to be assigned to the variables $z, \vec{w}$, if $\langle M, \in \rangle \vDash \forall x \in z\, \exists! y\, \varphi[a, \vec{c}]$ then there is $b \in M$ such that $\langle M, \in \rangle \vDash \forall x \in z\, \exists y \in v\, \varphi[a, \vec{c}, b]$, with b assigned to the variable v. Then $\varphi, a, \vec{c}$ yield a function $f\colon a \to M$, and by case assumption there is $b \in M$ such that $\operatorname{ran} f \subseteq b$. This is the b we were looking for.

$\langle M, \in \rangle \vDash$ AC if and only if $\forall \mathcal{A} \in M\, (\forall A \in \mathcal{A}\, (A \neq \emptyset) \Rightarrow \exists f \in M\, (f$ is a choice function for $\mathcal{A}))$.

The result follows from the straightforward verification that $\varphi(f, x)$ saying "$x \neq \emptyset$, every element of x is non-empty, and $f\colon x \to \bigcup x$ is a choice function" is Δ_0.

Proof of Theorem 19.15

All axioms of ZFC except the axiom of infinity hold in V_ω.

It is enough to check that replacement and choice hold in V_ω. As we shall see (Exercise 21.52), every V_n is finite, hence every element of V_ω is finite. It follows that every $x \in \mathrm{V}_\omega$ is well-orderable, hence AC holds by Theorem 18.3. Moreover, if $A \in \mathrm{V}_\omega$ and $F \colon A \to \mathrm{V}_\omega$, then $F``A$ is finite, $F``A = \{a_0, \ldots, a_{n-1}\}$. For every $i < n$, let $m_i < \omega$ be such that $a_i \in \mathrm{V}_{m_i}$. Then $F``A \subseteq \mathrm{V}_m$, where $m = \max\{m_0, \ldots, m_{n-1}\}$, hence $F``A \in \mathrm{V}_{m+1}$.

All axioms of ZF except possibly for replacement hold in V_λ, if $\lambda > \omega$ is limit.

Since $\omega \in \mathrm{V}_\lambda$ we apply Theorem 19.22.

Assuming choice, then AC holds in V_λ, if λ is limit.

If $\mathcal{A} \in \mathrm{V}_\lambda$ is a non-empty family of non-empty sets, by AC there is a choice function $f\colon \mathcal{A} \to \bigcup \mathcal{A}$. If $\alpha < \lambda$ is such that $\mathcal{A} \in \mathrm{V}_{\alpha \dotplus 1}$ then $f \in \mathrm{V}_{\alpha \dotplus 3}$ so we are done by Theorem 19.22.