Elements of Mathematical Logic

Section 20 of Chapter V

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Cardinal exponentiation

Cardinal exponentiation is defined by

$$\lambda^\kappa = \left|{}^\kappa\lambda\right|.$$

This definition requires that the set ${}^\kappa\lambda$ be well-ordered, so cardinal exponentiation is defined assuming AC.

$$\begin{aligned}
\kappa^\lambda &\leq \nu^\mu && \text{if } \kappa \leq \nu \text{ and } \lambda \leq \mu \\
\left(\kappa^\lambda\right)^\mu &= \kappa^{\lambda\cdot\mu} \\
\kappa^{\lambda+\mu} &= \kappa^\lambda \cdot \kappa^\mu \\
(\kappa\cdot\lambda)^\mu &= \kappa^\mu \cdot \lambda^\mu.
\end{aligned}$$

Definition 20.1

If X is a set and κ is a cardinal $\mathscr{P}_\kappa(X) = \{Y \subseteq X \mid |Y| < \kappa\}$ is the collection of all well-orderable subsets of X of size less than κ.

Note that $\mathscr{P}_\kappa(\lambda) = [\lambda]^{<\kappa}$ has size $\lambda^{<\kappa} \leq \lambda^\kappa$, where

$$\lambda^{<\kappa} \stackrel{\text{def}}{=} \sup \{\lambda^\nu \mid \nu \in \mathrm{Card} \wedge \nu < \kappa\} .$$

The class-function $\beth \colon \mathrm{Ord} \to \mathrm{Card}$ is defined by recursion by

$$\beth_0 = \omega, \qquad \beth_{\alpha+1} = 2^{\beth_\alpha}, \qquad \beth_\lambda = \sup_{\alpha<\lambda} 2^{\beth_\alpha}, \text{ for } \lambda \text{ limit.}$$

The continuum hypothesis

CH is $2^{\aleph_0} = \aleph_1$, or, equivalently, $\forall X \subseteq \mathbb{R}\,(|X| \leq \aleph_0 \vee |X| = |\mathbb{R}|)$.
The **generalized continuum hypothesis** GCH is
$\forall \alpha \in \mathrm{Ord}\left(2^{\aleph_\alpha} = \aleph_{\alpha+1}\right)$, or, equivalently,
$\forall X \subseteq \mathscr{P}(\aleph_\alpha)\,(|X| \leq \aleph_\alpha \vee |X| = |\mathscr{P}(\aleph_\alpha)|)$.
In the absence of AC, the continuum hypothesis is stated as follows:

$$\forall \mathcal{A} \subseteq \mathscr{P}(\omega)\big(\mathcal{A} \precsim \omega \,\vee\, \mathcal{A} \asymp \mathscr{P}(\omega)\big).$$

The generalized continuum hypothesis becomes

$$\forall X\, \forall \mathcal{A} \subseteq \mathscr{P}(X)\,\big(\mathcal{A} \precsim X \vee \mathcal{A} \asymp \mathscr{P}(X)\big)$$

and stated this way, the generalized continuum hypothesis implies AC.

Which sets are well-orderable?

$\mathsf{AC}_I(X)$ says: If $X \neq \emptyset$ is a set, then for every $\langle A_i \mid i \in I \rangle$ such that $\forall i \in I\ (\emptyset \neq A_i \subseteq X)$, there is $\langle a_i \mid i \in I \rangle$ such that $\forall i \in I\ (a_i \in A_i)$. Let AC_I be $\forall X\, \mathsf{AC}_I(X)$, let $\mathsf{AC}(X)$ be $\forall I\, \mathsf{AC}_I(X)$. Thus

$$\mathsf{AC} \Leftrightarrow \forall I\, \forall X\, \mathsf{AC}_I(X).$$

If $X \twoheadrightarrow Y$ and $J \rightarrowtail I$, then $\mathsf{AC}_I(X) \Rightarrow \mathsf{AC}_J(Y)$
If X is well-orderable then $\mathsf{AC}(X)$ (Theorem 18.3).

AC

Theorem 20.3

AC(X) implies that X is well-orderable.

Proof.

Without loss of generality $X \neq \emptyset$ and fix a choice function C for X. We will construct a bijection from some $\bar{\alpha} < \mathrm{Hrtg}(X)$ onto X.
Let us give a sketch of the proof: let x_0 be an element of X, for example $x_0 = C(X)$, and suppose we have constructed $x_0, x_1, \ldots, x_\beta, \ldots$, distinct elements of X, with $\beta < \alpha$. If $X = \{x_\beta \mid \beta < \alpha\}$ then $\alpha \to X$, $\beta \mapsto x_\beta$ is the required bijection. Otherwise pick a new element $x_\alpha \in X$ different from the previous ones, for example $x_\alpha = C(X \setminus \{x_\beta \mid \beta < \alpha\})$. If the map $\alpha \mapsto x_\alpha$ were defined for all $\alpha < \mathrm{Hrtg}(X)$, then we would have an injective map $\mathrm{Hrtg}(X) \rightarrowtail X$, a contradiction. Therefore there is $\bar{\alpha} < \mathrm{Hrtg}(X)$ such that $X = \{x_\beta \mid \beta < \bar{\alpha}\}$. □

Zorn's lemma and its relatives

$\textsc{Zorn}(X)$

If $\leq$ is an ordering on X such that every chain has an upper bound, then $\exists x \in X\,(x \text{ is maximal})$.

Zorn's Lemma is $\forall X\, \textsc{Zorn}(X)$.

If the assumption "every chain" is strengthened to "every upward directed set" a weaker statement is obtained: $\textsc{wZorn}(X)$. The **weak Zorn's Lemma** asserts that $\forall X\, \textsc{wZorn}(X)$.

$\textsc{MaxHaus}(X)$

If $\leq$ is an ordering on X, then $\exists C \subseteq X\,(C \text{ is a maximal chain})$

Hausdorff's maximality principle is $\forall X\, \textsc{MaxHaus}(X)$.

Zorn's lemma and its relatives

Proposition 20.5

Let $X \neq \emptyset$ be a set:
X is well-orderable $\Rightarrow \textsc{MaxHaus}(X) \Rightarrow \textsc{Zorn}(X) \Rightarrow \textsc{wZorn}(X)$ and $\textsc{wZorn}(\mathscr{P}(X \times X)) \Rightarrow X$ is well-orderable.

X is well-orderable implies $\textsc{MaxHaus}(X)$

Assume X is is well-orderable and, towards a contradiction, let $\leq$ be an ordering on X without maximal chains. For $C \subseteq X$ a chain, the set $K(C) = \{x \in X \setminus C \mid C \cup \{x\} \text{ is a chain}\}$ is nonempty. Fix a choice function $F \colon \mathscr{P}(X) \setminus \{\emptyset\} \to X$. Then

$$g \colon \mathrm{Hrtg}(X) \to X, \quad \alpha \mapsto F\left(K\left(\{g(\beta) \mid \beta < \alpha\}\right)\right).$$

is injective: a contradiction!

$\textsc{MaxHaus}(X)$ implies $\textsc{Zorn}(X)$

Let $\leq$ be an ordering on X such that every chain has an upper bound. If $C \subseteq X$ is a maximal chain, then the upper bound of C belongs to C so it is a maximal element of X.

$\textsc{Zorn}(X)$ implies $\textsc{wZorn}(X)$

Trivial.

$\textsc{wZorn}(\mathscr{P}(X \times X))$ implies X is well-orderable

Let $\mathcal{P} \subseteq \mathscr{P}(X \times X)$ be the set of all well-orders R such that $\operatorname{fld}(R) \subseteq X$. If $R, S \in \mathcal{P}$ set
$R \trianglelefteq S \Leftrightarrow \exists a \in \operatorname{fld}(S)[\operatorname{fld}(R) = \operatorname{pred}(a; S) \wedge R = S \cap \operatorname{fld}(R)^2]$. By $\textsc{wZorn}(\mathscr{P}(X \times X))$ there is a maximal $\bar{R} \in \mathcal{P}$. Towards a contradiction suppose $\operatorname{fld}(\bar{R}) \neq X$ and fix $a \in X \setminus \operatorname{fld}(\bar{R})$. Consider
$S = R \cup \{(y, a) \mid y \in \operatorname{fld}(\bar{R})\} \cup \{(a, a)\}$. Then $S \in \mathcal{P}$ e $\bar{R} \triangleleft S$, against the maximality of $\bar{R}$. Therefore $\bar{R}$ well-orders X.

Theorem 20.6

AC is equivalent to $\forall\alpha \in \mathrm{Ord}\,(\mathscr{P}(\alpha)$ is well-orderable).

Proof.

It is enough to show that each V_α is well-orderable.
If V_α is well-orderable, fix a bijection $f \colon \mathrm{V}_\alpha \to \gamma$, and by case assumption there is a well-order on $\mathscr{P}(\gamma)$ inducing via f a well-ordering on $\mathrm{V}_{\alpha+1} = \mathscr{P}(\mathrm{V}_\alpha)$. If λ is limit and if we can pick (without AC!) a well-order $\lhd_\alpha$ on V_α, for all $\alpha < \lambda$, then for $x, y \in \mathrm{V}_\lambda$

$$x \lhd_\lambda y \Leftrightarrow \exists\alpha < \lambda \left[(x \in \mathrm{V}_\alpha \wedge y \notin \mathrm{V}_\alpha) \vee (x, y \in \mathrm{V}_{\alpha+1} \setminus \mathrm{V}_\alpha \wedge x \lhd_{\alpha+1} y)\right] \quad (*)$$

is a well-order on V_λ. Let $\gamma = \sup_{\alpha<\lambda} \gamma_\alpha^+$ where $\gamma_\alpha = |\mathrm{V}_\alpha|$, and let $\prec$ be a well-order of $\mathscr{P}(\gamma)$. Set $\lhd_0 = \emptyset$; if $\lhd_\alpha$ is a well-order on V_α and $f_\alpha \colon \mathrm{V}_\alpha \to \gamma^+$ is its enumerating function then define $\lhd_{\alpha+1}$ on $\mathrm{V}_{\alpha+1}$ via f_α and $\prec$; if $\nu < \lambda$ is a limit ordinal apply the construction (*) with ν instead of λ. □

Theorem 20.7. The following are equivalent:

1. AC.
2. Hausdorff's maximality principle.
3. Zorn's Lemma.
4. The weak Zorn's Lemma.
5. **Teichmüller-Tukey's Lemma:** Let $\emptyset \neq \mathcal{F} \subseteq \mathscr{P}(X)$ be a family of **finite character**, that is $\forall Y \subseteq X\ (Y \in \mathcal{F} \Leftrightarrow \forall Z \subseteq Y\ (Z \text{ finite} \Rightarrow Z \in \mathcal{F}))$. Then any $Y \in \mathcal{F}$ is contained in a maximal $Z \in \mathcal{F}$.
6. The **Axiom of Multiple Choices (**AMC**):** For any set $X \neq \emptyset$ there is a function $F \colon \mathscr{P}(X) \setminus \{\emptyset\} \to \mathscr{P}(X) \setminus \{\emptyset\}$ such that $F(A) \subseteq A$ is finite, for all $\emptyset \neq A \subseteq X$.
7. Every preorder contains a maximal free subset.
8. **Kurepa's maximality principle:** Every order contains a maximal free subset.
9. Every linear order can be well-ordered.

Proof

Proposition 20.5 says that AC, Hausdorff, Zorn and weak-Zorn are equivalent, that is ❶ $\Leftrightarrow$ ❷ $\Leftrightarrow$ ❸ $\Leftrightarrow$ ❹ .
Every order is a preorder so ❼ $\Rightarrow$ ❽ .
❶ $\Rightarrow$ ❻ (i.e. multiple choices) is trivial.
Let us prove that ❹ $\Rightarrow$ ❺ $\Rightarrow$ ❼ , that ❻ $\vee$ ❽ $\Rightarrow$ ❾ and that ❾ $\Rightarrow$ ❶ .

❾ $\Rightarrow$ ❶ 'every linear order is well-orderable' implies AC

${}^{\alpha}2$ is linearly ordered by $<_{\mathrm{lex}}$, so ${}^{\alpha}2$ is well-orderable. As ${}^{\alpha}2 \asymp \mathscr{P}(\alpha)$ the result follows from Theorem 20.6.

❹ $\Rightarrow$ ❺ wZorn implies Teichmüller-Tuckey

Let $\mathcal{F} \subseteq \mathscr{P}(X)$ be of finite character, and let $Y \in \mathcal{F}$. If $\mathcal{D} \subseteq \mathcal{F}$ is an upward directed collection of sets containing Y, then $\bigcup \mathcal{D} \in \mathcal{F}$ by the finite character of $\mathcal{F}$, so $\mathcal{D}$ has an upper bound in $\mathcal{F}$. Therefore there is a $Z \in \mathcal{F}$ which is maximal and contains Y.

Proof

❺ ⇒ ❼ Teichmüller-Tuckey implies that every preorder has a maximal free subset

Let $\langle X, \leq \rangle$ be a preordered set. The family $\mathcal{F}$ of all free subsets of X has finite character, and $\emptyset \in \mathcal{F}$, so it contains a maximal set.

❻ ⇒ ❾ ‘multiple choices’ implies ‘every linear order is well-orderable’

Let $\langle X, \leq \rangle$ be a linear order. By assumption there is $G \colon \mathscr{P}(X) \setminus \{\emptyset\} \to \mathscr{P}(X) \setminus \{\emptyset\}$ such that $G(A) \subseteq A$ is finite, for all $\emptyset \neq A \subseteq X$. Let $g(A)$ be the minimum of $G(A)$. Then g is a choice function for X.

Proof

8 ⇒ 9 'Kurepa's maximality principle' implies 'every linear order is well-orderable'

Let $\langle X, \leq \rangle$ be a linear order. We prove that there is a choice function for X and the result follows from Theorem 20.3.
Let $\preceq$ be the ordering on $\mathcal{P} = \{(A, a) \mid A \subseteq X \wedge a \in A\}$ defined by

$$(A, a) \preceq (B, b) \Leftrightarrow A = B \wedge a \leq b.$$

By assumption there is a maximal free $\mathcal{A} \subseteq \mathcal{P}$. It easy to check that $\mathcal{A}$ is a choice function for X.

Cardinality without choice

Given E a non-regular equivalence relation on a proper class $\mathcal{A}$, we would like a choice function $\boldsymbol{C}\colon \mathcal{A} \to \mathcal{A}$ such that

$$\forall x \in \mathcal{A}\,(\boldsymbol{C}(x) \in [x]_E)$$

and

$$\forall x, y \in \mathcal{A}\,(x \mathrel{E} y \Rightarrow \boldsymbol{C}(x) = \boldsymbol{C}(y))\,.$$

The existence of such $\boldsymbol{C}$ is equivalent to the existence of a **transversal** T **for the relation** E, that is: a class $T \subseteq \mathcal{A}$ such that $T \cap [x]_E$ is a singleton for all $x \in \mathcal{A}$.

Example

If $\mathcal{A}$ is the class of all well-ordered sets and E is the isomorphism relation, then every equivalence class contains exactly one ordinal, so we may assume that $\boldsymbol{C}(A,<) = \operatorname{ot}(A,<)$;

Example

If $\mathcal{A}$ is the class of all finitely generated abelian groups and E is the isomorphism relation, then let $\boldsymbol{C}(G)$ be the unique

$$\mathbb{Z}^n \times \mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k\mathbb{Z}$$

isomorphic to G, where $n \geq 0$ and $p_1 \leq p_2 \leq \cdots \leq p_k$ primes and $k \geq 0$.

Example

Assuming AC and $\mathcal{A} = \mathrm{V}$ and E the relation $\asymp$, then $\boldsymbol{C}(A) =$ the unique cardinal κ in bijection with A.

For any equivalence relation E, one can define $[\![\cdot]\!]_E \colon \mathcal{A} \to \mathrm{V}$ such that

$$\emptyset \neq [\![x]\!]_E \subseteq [x]_E \quad \text{and} \quad x \mathrel{E} y \Leftrightarrow [\![x]\!]_E = [\![y]\!]_E.$$

The set $[\![x]\!]_E$ is the **Scott E-equivalence class**, defined by

$$[\![x]\!]_E = \{y \mid y \mathrel{E} x \wedge \forall z \, (z \mathrel{E} x \Rightarrow \operatorname{rank}(x) \leq \operatorname{rank}(z))\}$$

or equivalently: $[\![x]\!]_E = [x]_E \cap \mathrm{V}_{\bar{\alpha}}$, where $\bar{\alpha} = \min \{\alpha \mid \mathrm{V}_\alpha \cap [x]_E \neq \emptyset\}$.

The **order type** of an ordered set $\langle A, <\rangle$ is

$$\mathrm{type}\langle A, <\rangle = \begin{cases} \mathrm{ot}\langle A, <\rangle & \text{if } \langle A, <\rangle \text{ is a well-order,} \\ [\![(A, <)]\!]_{\cong} & \text{otherwise,} \end{cases}$$

where $\cong$ is the isomorphism relation.
In absence of AC the **cardinality** of a set X is

$$\mathrm{card}(X) = \begin{cases} |X| & \text{if } X \text{ is well-orderable,} \\ [\![X]\!]_{\asymp} & \text{otherwise.} \end{cases}$$

Cardinalities are denoted by lower case german letters $\mathfrak{a}, \mathfrak{a}, \ldots$. The ordering on cardinalities is given by

$$\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow A \precsim B \text{ for some/every } A \in \mathfrak{a} \text{ and } B \in \mathfrak{b}.$$

Thus $\mathfrak{a} = \mathfrak{b} \Leftrightarrow A \asymp B$ and $\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow A \precsim B$.

Theorem 20.10

AC is equivalent to $\forall \mathfrak{a}, \mathfrak{b}\ (\mathfrak{a} \leq \mathfrak{b}\ \vee\ \mathfrak{b} \leq \mathfrak{a})$, that is $\forall A, B\,(A \precsim B\ \vee\ B \precsim A)$.

Proof.

It is enough to show that if $A \precsim B \vee B \precsim A$ for all sets A, B, then every set is well-orderable. Let A be a set: since $\mathrm{Hrtg}(A) \not\precsim A$ by Hartog's Theorem, then $A \precsim \mathrm{Hrtg}(A) \subseteq \mathrm{Ord}$ so A is well-orderable. □

Addition and product of cardinalities are defined as follows:

$$\mathfrak{a} + \mathfrak{b} = \mathrm{card}(A \uplus B), \qquad \mathfrak{a} \cdot \mathfrak{b} = \mathrm{card}(A \times B),$$

for some/any $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$.

If A and B have at least two elements,
$\mathrm{card}(A) + \mathrm{card}(B) \leq \mathrm{card}(A) \cdot \mathrm{card}(B)$.
Assuming AC one has that if A is infinite, then $A \asymp A \times A$; thus if A and B are infinite disjoint sets then

$$A \cup B \asymp (A \cup B) \times (A \cup B) \asymp A \cup (A \times B) \cup (B \times A) \cup B,$$

and hence $A \times B \rightarrowtail A \cup B$, that is
$\mathrm{card}(A) \cdot \mathrm{card}(B) \leq \mathrm{card}(A) + \mathrm{card}(B)$. Thus $\forall A\, (A \asymp A \times A)$ implies

$$\forall A, B\, (A, B \text{ infinite } \Rightarrow \mathrm{card}(A) + \mathrm{card}(B) = \mathrm{card}(A) \cdot \mathrm{card}(B).$$

Theorem 20.11

The following are equivalent:

1. AC,
2. $\forall \mathfrak{a}(\mathfrak{a} \not< \omega \Rightarrow \mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a})$,
3. $\forall \mathfrak{a} \forall \mathfrak{b}(\mathfrak{a}, \mathfrak{b} \not< \omega \Rightarrow \mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a} + \mathfrak{b})$.

Proof.

1 $\Rightarrow$ 2 is clear, and 2 $\Rightarrow$ 3 follows from what we proved before.
We show that 3 implies that every set A is well-orderable. We may assume that A is disjoint from $B = \mathrm{Hrtg}(A)$. By assumption there is a bijection $F\colon A \times \mathrm{Hrtg}(A) \to A \cup \mathrm{Hrtg}(A)$. Since $\mathrm{Hrtg}(A) \precsim A$ is impossible,

$$\forall x \in A \, \exists \alpha \in \mathrm{Hrtg}(A) \, (F(x, \alpha) \notin A).$$

If $\alpha(x)$ is the least witness, then $A \to \mathrm{Hrtg}(A)$, $x \mapsto F(x, \alpha(x))$, is injective hence A is well-orderable. □

Weak forms of the axiom of choice

Recall from Section 14.E the Axiom of Countable Choices AC_ω, that is $\forall X\, \mathsf{AC}_\omega(X)$, where

$\mathsf{AC}_\omega(X)$

$$\forall \langle A_i \mid i \in \omega \rangle \in {}^{\omega}(\mathscr{P}(X) \setminus \{\emptyset\})\, \exists \langle a_i \mid i \in \omega \rangle \in {}^{\omega}X\, \forall i \in \omega\, (a_i \in A_i)$$

Recall:

Theorem 14.31

AC_ω implies that the countable union of countable sets is countable.

Theorem 20.12

Assume $\mathsf{AC}_\omega(\mathbb{R})$. Then ω_1 is not a countable union of countable sets. In particular: if $\alpha_n < \omega_1$ for all $n < \omega$, then $\sup_n \alpha_n < \omega_1$.

Proof.

Let $X_n \subseteq \omega_1$ be countable sets, for $n < \omega$. Without loss of generality, we may assume that each X_n is infinite, and let

$$A_n = \{R \subseteq \omega \times \omega \mid R \text{ is a well-order of } \omega \text{ and } \operatorname{ot}\langle \omega, R\rangle = \operatorname{ot}\langle X_n, \leq\rangle\}.$$

As $\mathscr{P}(\omega \times \omega) \asymp \mathbb{R}$ we may choose $R_n \in A_n$ for all $n \in \omega$, and let $f_n \colon \langle \omega, R_n\rangle \to \langle X_n, \leq\rangle$ be the unique isomorphism. Then

$$\omega \times \omega \to \bigcup_{n\in\omega} X_n, \quad (n, m) \mapsto f_n(m)$$

is surjective, thus $\bigcup_{n\in\omega} X_n$ is countable. □

Corollary 20.13

$AC_\omega(\mathbb{R})$ implies that $\mathbb{R}$ is not a countable union of countable sets.

DC

For any set $X \neq \emptyset$, $DC(X)$ is: If R is a relation on X such that $\forall x \exists y \, (x \mathrel{R} y)$, then for any $x_0 \in X$ there is an $f \in {}^\omega X$ such that $f(0) = x_0$ and $\forall n \, (f(n) \mathrel{R} f(n+1))$.

If X is well-orderable, then $DC(X)$ is provable, and if $X \twoheadrightarrow Y$ then $DC(X) \Rightarrow DC(Y)$.

Proposition 20.14

For X a nonempty set:

- $AC(X) \Rightarrow DC(X)$,
- $DC(X \times \omega) \Rightarrow AC_\omega(X)$.

In particular: $AC \Rightarrow DC \Rightarrow AC_\omega$.

Proof.

Assume AC(X) towards proving DC(X). Let $X \neq \emptyset$ and let $R \subseteq X \times X$ be such that $\forall x \exists y\, (x \mathrel{R} y)$. Pick $x_0 \in X$ and a choice function $C \colon \mathscr{P}(X) \setminus \{\emptyset\} \to X$. Define $f \colon \omega \to X$ by $f(0) = x_0$ and $f(n+1) = C\left(\{y \in X \mid f(n) \mathrel{R} y\}\right)$. Then f witnesses DC(X) for R, x_0.
Assume DC$(X \times \omega)$ towards proving AC$_\omega(X)$. Given $\{A_n \mid n \in \omega\} \subseteq \mathscr{P}(X) \setminus \{\emptyset\}$ let R be the relation on $X \times \omega$ defined by

$$(a,n) \mathrel{R} (b,m) \Leftrightarrow m = n+1 \wedge (a \in A_n \Rightarrow b \in A_m).$$

For every $(a,n) \in X \times \omega$ there is some $b \in X$ such that $(a,n) \mathrel{R} (b,n+1)$: if $a \in A_n$ pick $b \in A_{n+1}$, if $a \notin A_n$ let $b = a$. Fix an element $a_0 \in A_0$: by DC$(X \times \omega)$ there is a function $f \colon \omega \to X \times \omega$ such that $f(0) = (a_0, 0)$ and $f(n) \mathrel{R} f(n+1)$ for all n. The function $g \colon \omega \to X$

$$g(n) = \text{the first component of the ordered pair } f(n)$$

is the required function. □