Elements of Mathematical Logic

Section 21 of Chapter V

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Definition 21.1 (AC)

Let $\langle \kappa_i \mid i \in I \rangle$ be a sequence of cardinals.

- The **generalized sum** of the κ_is is $\sum_{i \in I} \kappa_i = |\bigcup_{i \in I} \{i\} \times \kappa_i|$;
- The **generalized product** of the κ_is is $\prod_{i \in I} \kappa_i = |\times_{i \in I} \kappa_i|$.

- $\kappa = \sum_{i \in \kappa} 1 = \sum_{i \in \kappa} \kappa_i$, with $\kappa_i = 1$,
- $2^\kappa = \prod_{i \in \kappa} 2 = \prod_{i \in \kappa} \kappa_i$, with $\kappa_i = 2$,
- the operations of generalized sum and product are monotone, that is if $\kappa_i \leq \lambda_i$, then $\sum_{i \in I} \kappa_i \leq \sum_{i \in I} \lambda_i$.

Proposition 21.2

If I is a well-orderable set and $1 \leq \kappa_i$ for every $i \in I$, then

$$\sum_{i \in I} \kappa_i \leq |I| \cdot \sup_{i \in I} \kappa_i,$$

and if $\max(|I|, \sup_{i \in I} \kappa_i) \geq \omega$, then equality holds.

Proof.

The inclusion $\bigcup_{i \in I} \{i\} \times \kappa_i \subseteq I \times \sup_{i \in I} \kappa_i$ proves the inequality. For every $\alpha \in \sup_{i \in I} \kappa_i$ pick $i(\alpha) \in I$ such that $\alpha \in \kappa_{i(\alpha)}$: the function $\sup_{i \in I} \kappa_i \to \bigcup_{i \in I} \{i\} \times \kappa_i$, $\alpha \mapsto (i(\alpha), \alpha)$ is injective and proves that $\sup_{i \in I} \kappa_i \leq \sum_{i \in I} \kappa_i$. By monotonicity $|I| = \sum_{i \in I} 1 \leq \sum_{i \in I} \kappa_i$. Therefore $\max(|I|, \sup_{i \in I} \kappa_i) \leq \sum_{i \in I} \kappa_i$. The conclusion follows from Corollary 18.29. $\square$

Theorem 21.3 (AC)

If I and $\{X_i \mid i \in I\}$ are sets, then $|\bigcup_{i\in I} X_i| \leq |I| \cdot \sup_{i\in I}|X_i|$.

Proof.

For each $i \in I$ choose a bijection $f_i \colon X_i \to |X_i|$ and for each $x \in \bigcup_{i\in I} X_i$ choose $i(x) \in I$ such that $x \in X_{i(x)}$. The function

$$\bigcup_{i\in I} X_i \to \bigcup_{i\in I}\{i\} \times |X_i| \quad x \mapsto (i(x), f_{i(x)}(x))$$

is injective hence $|\bigcup_{i\in I} X_i| \leq \sum_{i\in I}|X_i|$. The result follows immediately from Proposition 21.2. $\square$

If $I \neq \emptyset$ and $\kappa_i \leq \lambda_i \geq 2$ then $\sum_{i\in I} \kappa_i \leq \prod_{i\in I} \lambda_i$.

Theorem 21.4 J. König

Assume AC. If $\kappa_i < \lambda_i$ for all $i \in I$, then

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i .$$

Proof.

It is enough to show that $\sum_{i \in I} \kappa_i \not\geq \prod_{i \in I} \lambda_i$, that is no $F \colon \bigcup_i \{i\} \times \kappa_i \to \bigtimes_{i \in I} \lambda_i$ can be surjective. Fix such an F: for every $i \in I$, the set $\{F(i, \alpha)(i) \mid \alpha \in \kappa_i\}$ has cardinality $< \lambda_i$, so we can define a function $f \in \bigtimes_{i \in I} \lambda_i$:

$$f(i) = \min\left(\lambda_i \setminus \{F(i, \alpha)(i) \mid \alpha \in \kappa_i\}\right).$$

Let us check that $f \notin \operatorname{ran}(F)$: if, towards a contradiction, $f = F(i_0, \alpha_0)$ for some i_0, α_0, then $f(i_0) \notin \{F(i_0, \alpha)(i_0) \mid \alpha \in \kappa_{i_0}\}$ by definition of f, a contradiction. $\square$

Definition 21.5

A function $f\colon \beta \to \alpha$ is **cofinal (in α)** if $\operatorname{ran}(f)$ is unbounded in α, that is $\forall \alpha' < \alpha\, \exists \beta' < \beta\, (\alpha' \leq f(\beta'))$. The **cofinality** of an ordinal α is the least β such that there is a cofinal $f\colon \beta \to \alpha$. This β is denoted by $\operatorname{cof}(\alpha)$.

Example

- $\mathrm{id} \restriction \alpha\colon \alpha \to \alpha$ is cofinal, so $\operatorname{cof}(\alpha) \leq \alpha$. In particular $\operatorname{cof}(0) = 0$.
- The cofinality of $\gamma + 1$ is 1, as witnessed by $0 \mapsto \gamma$. Conversely, if λ is limit, $\operatorname{cof}(\lambda)$ is limit.
- $\operatorname{cof}(\omega) = \omega$ and (assuming a bit of choice) $\operatorname{cof}(\omega_1) = \omega_1$. On the other hand, $\operatorname{cof}(\aleph_\omega) = \omega$, since $n \mapsto \aleph_n$ is cofinal.

A cofinal map need not be monotone, but...

Lemma 21.7

There is a cofinal monotone function $f\colon \operatorname{cof}(\alpha) \to \alpha$.

Proof.

Let $g\colon \operatorname{cof}(\alpha) \to \alpha$ be cofinal, and to avoid trivialities we may assume that α is limit. For $\beta < \operatorname{cof}(\alpha)$ let $f(\beta) = \max\left(g(\beta), \sup_{\gamma<\beta} f(\gamma)\right)$. By construction f is monotone and cofinal. If there is a least $\bar{\beta} < \operatorname{cof}(\alpha)$ such that $\sup_{\gamma<\bar{\beta}} f(\gamma) = \alpha$, then $f\colon \bar{\beta} \to \alpha$ would be cofinal: a contradiction. Therefore $f\colon \operatorname{cof}(\alpha) \to \alpha$ is as required. □

Lemma 21.7

If $f\colon \beta \to \alpha$ and $g\colon \gamma \to \beta$ are cofinal and f is also monotone, then $f \circ g\colon \gamma \to \alpha$ is cofinal.

Proof.

If $\alpha' < \alpha$ let $\beta' < \beta$ be such that $f(\beta') \geq \alpha'$ and let $\gamma' < \gamma$ be such that $g(\gamma') \geq \beta'$. Then $f(g(\gamma')) \geq \alpha'$. □

Corollary 21.8

$\operatorname{cof}(\operatorname{cof}(\alpha)) = \operatorname{cof}(\alpha)$.

Definition 21.9

A limit ordinal λ is **regular** if $\operatorname{cof}(\lambda) = \lambda$. Otherwise it is **singular**. If λ is an infinite cardinal, we will speak of **regular** or **singular cardinal**.

If $f \colon |\lambda| \to \lambda$ is a bijection, then f is cofinal, hence a regular ordinal is a cardinal. Conversely, limit ordinals that are not cardinals are singular.

Theorem 21.10 (AC)

If $\kappa \geq \omega$ then κ^+ is regular.

Proof.

Towards a contradiction suppose $\operatorname{cof}(\kappa^+) \leq \kappa$. Let $f \colon \operatorname{cof}(\kappa^+) \to \kappa^+$ be cofinal. Then $\kappa^+ = \bigcup_{i < \operatorname{cof}(\kappa^+)} f(i)$ hence

$$\kappa^+ \leq \textstyle\sum_{i < \operatorname{cof}(\kappa^+)} |f(i)| \leq \operatorname{cof}(\kappa^+) \cdot \sup_{i < \operatorname{cof}(\kappa^+)} |f(i)| \leq \kappa,$$

a contradiction. □

Theorem 21.11 (AC)

If κ is a singular cardinal, then there is an increasing sequence of regular cardinals $\langle \kappa_i \mid i < \operatorname{cof}(\kappa) \rangle$ such that

$$\kappa = \sup_{i < \operatorname{cof}(\kappa)} \kappa_i = \sum_{i < \operatorname{cof}(\kappa)} \kappa_i .$$

Proof.

Let $f \colon \operatorname{cof}(\kappa) \to \kappa$ be increasing and cofinal. The function

$$g(\alpha) = \min\{\lambda \in \kappa \mid \lambda \text{ is regular, } \lambda \geq f(\alpha) \text{ and } \forall \beta < \alpha \, (g(\beta) < \lambda)\}$$

is defined for all $\alpha < \operatorname{cof}(\kappa)$ since the regular cardinals are unbounded below κ hence if $\bar{\alpha} < \operatorname{cof}(\kappa)$ were the least ordinal such that $g(\bar{\alpha})$ is not defined, then it would mean that $\kappa = \sup_{\beta < \bar{\alpha}} g(\beta)$, that is $g \colon \bar{\alpha} \to \kappa$ would be cofinal, against $\bar{\alpha} < \operatorname{cof}(\kappa)$. Letting $\kappa_i = g(i)$, it follows that

$$\kappa = \sup_{i < \operatorname{cof}(\kappa)} \kappa_i \leq \sum_{i < \operatorname{cof}(\kappa)} \kappa_i \leq \kappa \cdot \operatorname{cof}(\kappa) = \kappa \qquad \square$$

Theorem 21.12 (AC)

$\kappa^{\operatorname{cof}(\kappa)} > \kappa$ when κ is an infinite cardinal.

Proof.

If κ is regular, the statement becomes $\kappa^\kappa = 2^\kappa > \kappa$, which is true by Cantor's Theorem. We may therefore suppose that $\operatorname{cof}(\kappa) < \kappa$. By Theorem 21.11 there are cardinals κ_i such that $\kappa = \sup_{i<\operatorname{cof}(\kappa)} \kappa_i$ and hence by König's Theorem 21.4

$$\kappa = \sum_{i<\operatorname{cof}(\kappa)} \kappa_i < \prod_{i<\operatorname{cof}(\kappa)} \kappa = \kappa^{\operatorname{cof}(\kappa)}.$$

□

Corollary 21.13 (AC)

$\mathrm{cof}(2^\kappa) > \kappa$ when κ is an infinite cardinal.

Proof.

If $\lambda = \mathrm{cof}(2^\kappa) \leq \kappa$, then $2^\kappa < (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\kappa$, a contradiction. □

In particular, $\mathrm{cof}(2^{\aleph_0}) > \aleph_0$ hence $2^{\aleph_0}$ can neither be $\aleph_\omega$, $\aleph_{\omega+\omega}$ (or, more generally, $\aleph_\lambda$ with $\lambda < \omega_1$ limit) nor can it be the least fixed point of the $\aleph$ function.

Hausdorff's formula

Theorem 21.14 (AC)

$\aleph_{\alpha+1}^{\aleph_\beta} = \max\left(\aleph_{\alpha+1}, \aleph_\alpha^{\aleph_\beta}\right)$.

Proof.

If $\aleph_{\alpha+1} \leq \aleph_\beta$ then by Proposition 18.30 $\aleph_\alpha^{\aleph_\beta} = \aleph_{\alpha+1}^{\aleph_\beta} > \aleph_\beta \geq \aleph_{\alpha+1}$ hence the result is proved.

Suppose instead that $\aleph_\beta < \aleph_{\alpha+1}$. If $f \colon \aleph_\beta \to \aleph_{\alpha+1}$, then by regularity of $\aleph_{\alpha+1}$ (Theorem 21.10) there is a $\gamma < \aleph_{\alpha+1}$ such that $\operatorname{ran} f \subseteq \gamma$. Thus ${}^{\aleph_\beta}\aleph_{\alpha+1} = \bigcup_{\gamma<\aleph_{\alpha+1}} {}^{\aleph_\beta}\gamma$ and by Theorem 21.3

$$\aleph_{\alpha+1}^{\aleph_\beta} = |\bigcup_{\gamma<\aleph_{\alpha+1}} {}^{\aleph_\beta}\gamma| \leq \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}.$$

The other inequality is immediate. □

Theorem 21.15 (Bukovsky–Hechler)

Assume AC. If $\operatorname{cof}(2^{<\kappa}) > \kappa > \operatorname{cof}(\kappa)$ then $2^\kappa = 2^{<\kappa}$.

Proof.

Let $\langle \kappa_\alpha \mid \alpha < \operatorname{cof}(\kappa) \rangle$ be increasing and $\sup_{\alpha \in \operatorname{cof}(\kappa)} \kappa_\alpha = \kappa$. If $\forall \alpha \in \operatorname{cof}(\kappa)\, \exists \beta \in \operatorname{cof}(\kappa)\, (2^{\kappa_\alpha} < 2^{\kappa_\beta})$, then $\operatorname{cof}(2^{<\kappa}) = \operatorname{cof}(\kappa) < \kappa$, against our assumption. Therefore there is γ such that $2^{\kappa_\beta} = 2^{\kappa_\gamma}$ for all $\beta \geq \gamma$. We may assume that $\kappa_\gamma \geq \operatorname{cof}(\kappa)$. Then

$$2^\kappa = 2^{\sum_{\alpha \in \operatorname{cof}(\kappa)} \kappa_\alpha} = \prod_{\alpha \in \operatorname{cof}(\kappa)} 2^{\kappa_\alpha} \leq (2^{\kappa_\gamma})^{\operatorname{cof}(\kappa)} = 2^{\kappa_\gamma} = 2^{<\kappa}. \qquad \square$$

Operations

Recall that

- an operation on a set X is an $f\colon {}^n X \to X$ for some $n < \omega$,
- if $\mathcal{F}$ is a collection of operations on X and $Y \subseteq X$, then $\mathrm{Cl}_{\mathcal{F}} Y$, the closure of Y under $\mathcal{F}$, is the smallest subset of X containing Y and closed under each $f \in \mathcal{F}$.

$\mathrm{Cl}_{\mathcal{F}} Y = \bigcup_n Y_n$, where $Y_{n+1} = Y_n \cup \{f(\vec{a}) \mid \vec{a} \in Y_n^{<\omega} \wedge f \in \mathcal{F}\}$ and $Y_0 = Y$.

Definition 21.16

A **generalized operation** on X is a $f\colon {}^\alpha X \to X$ where $\alpha \in \mathrm{Ord}$ is the **arity of** f, written $\mathrm{ar}\, f$; when $\alpha \geq \omega$ we will speak of **infinitary operations**, while ordinary operations, i.e. when $\alpha < \omega$, are often called **finitary operations**.

If $\mathcal{F}$ is a collection of generalized operations on X and $Y \subseteq X$, then

$$\mathrm{Cl}_{\mathcal{F}}\, Y = \bigcap \{Z \subseteq X \mid Y \subseteq Z \wedge \forall f \in \mathcal{F}\, \forall \vec{a} \in {}^{\mathrm{ar}(f)}Z \ (f(\vec{a}) \in Z)\}$$

is the smallest subset of X containing Y and closed under each $f \in \mathcal{F}$.

Theorem 21.17

Let $\mathcal{F}$ be a family of generalized operations on a set X and let $Y \subseteq X$. Suppose λ is a regular cardinal such that $\lambda > \mathrm{ar}(f)$ for all $f \in \mathcal{F}$.

1. Then $\mathrm{Cl}_{\mathcal{F}}\, Y = \bigcup_{\beta<\lambda} Y_\beta$ where $Y_0 = Y$, $Y_\gamma = \bigcup_{\beta<\gamma} Y_\beta$ when γ is limit, and $Y_{\beta+1} = Y_\beta \cup \{f(\vec{a}) \mid f \in \mathcal{F} \wedge \vec{a} \in {}^{\mathrm{ar}(f)}Y_\beta\}$.
2. Assume AC and suppose $\kappa \geq \max(\lambda, |\mathcal{F}|, |Y|)$ and $\forall f \in \mathcal{F}\ \left(\kappa^{|\mathrm{ar}(f)|} = \kappa\right)$. Then $|\mathrm{Cl}_{\mathcal{F}} Y| \leq \kappa$.

$\mathcal{F}$ a family of generalized operations on X, $Y \subseteq X$, and
$\lambda = \operatorname{cof} \lambda > \operatorname{ar}(f)$ for all $f \in \mathcal{F}$.

$\operatorname{Cl}_{\mathcal{F}} Y = \bigcup_{\beta<\lambda} Y_\beta$ where $Y_0 = Y$, $Y_\gamma = \bigcup_{\beta<\gamma} Y_\beta$ when γ is limit, and
$Y_{\beta+1} = Y_\beta \cup \{f(\vec{a}) \mid f \in \mathcal{F} \wedge \vec{a} \in {}^{\operatorname{ar}(f)} Y_\beta\}$.

Proof.

$\overline{Y} = \bigcup_{\alpha<\lambda} Y_\alpha \subseteq \operatorname{Cl}_{\mathcal{F}} Y$ is clear.
For the other inclusion, if $f \in \mathcal{F}$ and $\alpha = \operatorname{ar} f$, then by regularity of λ every $\vec{a} \in {}^{\alpha}\overline{Y}$ belongs to some Y_β, so $f(\vec{a}) \in Y_{\beta+1} \subseteq \overline{Y}$. $\square$

$\mathcal{F}$ a family of generalized operations on X, $Y \subseteq X$, and $\lambda = \operatorname{cof}\lambda > \operatorname{ar}(f)$ for all $f \in \mathcal{F}$.

Assume AC and suppose $\kappa \geq \max(\lambda, |\mathcal{F}|, |Y|)$ and $\forall f \in \mathcal{F}\ \left(\kappa^{|\operatorname{ar}(f)|} = \kappa\right)$. Then $|\mathrm{Cl}_{\mathcal{F}} Y| \leq \kappa$.

Proof.

It is enough to show that $\forall \beta < \lambda\, (|Y_\beta| \leq \kappa)$. This is true if $\beta = 0$ or β limit. Suppose this holds for some β, so that $|Y_\beta| \leq \kappa$ and $|{}^{\operatorname{ar}(f)}Y_\beta| \leq \kappa$ for all $f \in \mathcal{F}$. As $\{f(\vec{a}) \mid f \in \mathcal{F} \wedge \vec{a} \in {}^{\operatorname{ar}(f)}Y_\beta\}$ is the surjective image of $\bigcup_{f \in \mathcal{F}} \{f\} \times {}^{\operatorname{ar}(f)}Y_\beta$, which has size $\leq |\mathcal{F}| \cdot \kappa$, then $|Y_{\beta+1}| \leq \kappa$. □

Theorem 21.18 (AC)

Let $\mathcal{F}$ is a family of generalized operations on a set X and let $Y \subseteq X$.

1. If $\mathrm{ar}(f) < \omega$ for all $f \in \mathcal{F}$, i.e. $\mathcal{F}$ is a family of finitary operations, then $|\mathrm{Cl}_{\mathcal{F}} Y| \leq \max(\omega, |\mathcal{F}|, |Y|)$.
2. If $\mathrm{ar}(f) < \omega_1$ for all $f \in \mathcal{F}$, and $|\mathcal{F}| \leq |Y|^{\omega}$, then $|\mathrm{Cl}_{\mathcal{F}} Y| \leq |Y|^{\omega}$.

Proof.

1. It is enough to check that $\lambda = \omega$ and $\kappa = \max(\omega, |\mathcal{F}|, |Y|)$ satisfy the hypotheses of Theorem 21.17, namely that $\kappa^n = \kappa$, which is immediate.
2. It is enough to observe that $\lambda = \omega_1$ and $\kappa = |Y|^{\omega}$ satisfy the hypotheses of Theorem 21.17, namely $\kappa^{\omega} = \kappa$. □

Example 21.19

If $\mathcal{M} = \langle M, \dots \rangle$ is an $\mathcal{L}$-structure, then the substructure generated by $Y \subseteq M$ has size $\leq \max(\omega, \lambda, |Y|)$, where λ is the cardinality of the set of non-logical symbols of $\mathcal{L}$.

Example 21.20

A Boolean algebra B is **countably complete** if it is closed under countable joins or, equivalently, countable meets. The smallest countably complete subalgebra of B containing $Y \subseteq B$ has size $\leq |Y|^\omega$.
A σ-**algebra** is an algebra of sets which is closed under countable unions or, equivalently, countable intersections; thus a σ-algebra is an example of a countably complete Boolean algebra. If X is a topological space, the σ-algebra generated by the open sets is the family $\mathrm{Bor}(X)$ of **Borel subsets** of X. By Section 13.G.4 when X is infinite, second countable, and T_1, then $|\mathrm{Bor}(X)| = 2^{\aleph_0}$.

Every ordinal is a topological space, and since α is a subspace of β when $\alpha < \beta$, the topology on an ordinal is induced by the topology on $\langle \mathrm{Ord}, \leq \rangle$.

Definition 21.21

Let $\Omega \leq \mathrm{Ord}$. A class $A \subseteq \Omega$ is **open** in Ω if for every $\alpha \in A$ there is are $\beta < \alpha < \gamma$ such that $(\beta; \gamma) \subseteq A$, with the proviso that if $\alpha = 0$ then we require $[0; \gamma) \subseteq A$ for some $\gamma > 0$. A class $C \subseteq \Omega$ is **closed** in Ω if $\Omega \setminus C$ is open in Ω; equivalently:

$$\forall \lambda \big(0 < \bigcup (C \cap \lambda) = \lambda \Rightarrow \lambda \in C \big).$$

Thus 0 and all successor ordinals are isolated points of Ω. The spaces $\omega \dotplus 1$ and $\omega \dotplus n$ are homeomorphic for all $1 \leq n < \omega$, while $\omega \dotplus 1$ and $\omega \dotplus \omega \dotplus 1$ are not homeomorphic, since the former has one non-isolated point, namely ω, while the latter has two non-isolated points, ω and $\omega \dotplus \omega$.

Proposition 21.22

An ordinal is a compact space if and only if it is either zero or else a successor ordinal.

Proof.

We will prove by induction on α that every open covering $\mathcal{U}$ of $\alpha \dot{+} 1$ has a finite subcovering. If $\alpha = 0$ the result follows at once, thus we may assume that $\alpha > 0$ and that $\beta \dot{+} 1$ be compact, for all $\beta < \alpha$. Let $\mathcal{U}$ be an open cover of $\alpha \dot{+} 1$ and let $U \in \mathcal{U}$ be such that $\alpha \in U$. Choose $\beta < \alpha$ such that $[\beta \dot{+} 1, \alpha] \subseteq U$: by inductive assumption there is a finite $\mathcal{U}_0 \subseteq \mathcal{U}$ covering $\beta \dot{+} 1 \leq \alpha$, hence $\mathcal{U}_0 \cup \{U\}$ is a finite open cover of $\alpha \dot{+} 1$.
Conversely, suppose λ is a limit ordinal: then $\{[0; \alpha) \mid \alpha < \lambda\}$ is an open covering of λ that has no finite subcovering. $\square$

Definition 21.23

A Hausdorff topological space is **totally disconnected** or **zero-dimensional** if every point has a neighborhood base made of clopen sets.
A topological space X is **completely regular** if given a closed set C and a point $x \notin C$ there is a continuous $f\colon X \to [0;1]$ such that $f(x) = 1$ and $\forall y \in C\,(f(y) = 0)$.

By Tietze's theorem, every metric space is completely regular, and a completely regular space is Hausdorff. An ordinal is a totally disconnected, completely regular space.

Proposition 21.24

Let X be a completely regular topological space that does not surject onto $\mathbb{R}$. Then X is totally disconnected.

Proof.

Fix $x \in X$ and V an open neighborhood, and let f be a continuous function such that $f(x) = 0$ and $f(y) = 1$ for all $y \in X \setminus V$. By assumption there is $r \in (0; 1) \setminus \operatorname{ran}(f)$. Then $f^{-1}[0; r] = f^{-1}[0; r)$ is a clopen neighborhood of x contained in V. $\square$

Corollary 21.25

A countable metric space is totally disconnected.

Every countable ordinal is homeomorphic to a countable closed subset of $\mathbb{R}$, hence by Proposition 21.22 every countable successor ordinal is homeomorphic to a countable compact subset of $\mathbb{R}$.
Which conditions must $f\colon \Omega \to \mathrm{Ord}$ satisfy in order to be continuous?
Continuity is never a problem on the successor ordinals, as they are isolated points. If $\gamma < \Omega$ is limit and $f(\gamma)$ is a successor, then by continuity of f, there is an interval $[\beta; \gamma]$ which is mapped by f in the singleton $\{f(\gamma)\}$; in other words: f is eventually constant below γ. If $\gamma < \Omega$ is limit and $f(\gamma)$ is limit, then for every $\delta < f(\gamma)$ there is $\beta < \gamma$ such that the interval $[\beta; \gamma]$ is mapped by f into the interval $[\delta; f(\gamma)]$. Therefore:

Lemma 21.26

Suppose $f\colon \Omega \to \mathrm{Ord}$ is monotone. Then f is continuous if and only if for every limit ordinal $\lambda < \Omega$

$f(\lambda) = \sup_{\beta<\lambda} f(\beta)$ and $\forall X \subseteq \lambda\, (\sup X = \lambda \Rightarrow f(\lambda) = \sup_{\nu \in X} f(\nu))$.

Thus if $f\colon \Omega \to \mathrm{Ord}$ is increasing and continuous, then $f(\lambda)$ is limit for all limit ordinals λ.

Proposition 21.27

Suppose Ω is either a regular cardinal or Ord. If $f\colon \Omega \to \Omega$ is increasing and continuous then $\operatorname{ran} f$ is closed and unbounded in Ω. Conversely, if C is closed and unbounded in Ω, then its enumerating function $f\colon \Omega \to C \subseteq \Omega$ is increasing and continuous.

Proof.

Suppose $f\colon \Omega \to \Omega$ is increasing and continuous. Then $f(\alpha) \geq \alpha$, as f is increasing, so $\operatorname{ran} f$ is unbounded in Ω. Suppose λ is limit and $\lambda \cap \operatorname{ran} f$ is unbounded in λ, and let $\nu = \{\alpha < \Omega \mid f(\alpha) < \lambda\}$; then ν is limit and $\lambda = f(\nu) \in \operatorname{ran} f$. Therefore $\operatorname{ran} f$ is closed in Ω.

Conversely, suppose C is closed and unbounded in Ω. The enumerating function $f\colon \Omega \to \Omega$ is increasing. If $\lambda \in \Omega$ is limit, then $\nu \stackrel{\text{def}}{=} \sup_{\gamma<\lambda} f(\gamma)$ is limit and $C = \operatorname{ran} f$ is unbounded in ν, so $\nu \in C$ and hence $f(\lambda) = \nu = \sup_{\gamma<\lambda} f(\gamma)$. Therefore f is continuous. □

In what follows κ is an uncountable regular cardinal. The next result shows that

$$\mathrm{Club}(\kappa) = \{X \subseteq \kappa \mid \exists C \subseteq X\,(C \text{ is closed and unbounded in } \kappa)\}$$

is a proper filter on κ. (Properness follows from the fact that $\emptyset$ is not unbounded, so if $X \in \mathrm{Club}(\kappa)$ then $\kappa \setminus X \notin \mathrm{Club}(\kappa)$.)

Theorem 21.28

If $C, D \subseteq \kappa$ are closed and unbounded in κ, then $C \cap D$ is closed and unbounded in κ.

Proof.

Clearly $C \cap D$ is closed, so it is enough to show that it is unbounded in κ. Given $\alpha < \kappa$ let us find a $\beta \in C \cap D$ with $\alpha < \beta$. Using that C and D are unbounded, let us construct inductively an increasing sequence of ordinals $\alpha < \gamma_0 < \delta_0 < \gamma_1 < \delta_1 < \ldots$ such that $\gamma_i \in C$ and $\delta_i \in D$. Let $\beta = \sup_i \gamma_i = \sup_i \delta_i$. Since κ is regular then $\beta \in \kappa$ and since C and D are closed, $\beta = \sup_i \gamma_i \in C$ and $\beta = \sup_i \delta_i \in D$, that is $\beta \in C \cap D$. $\square$

The assumption that κ be regular and uncountable cannot be removed—the sets $\{2n \mid n \in \omega\}$ and $\{2n+1 \mid n \in \omega\}$ are closed and unbounded in ω but their intersection $\emptyset$ is not unbounded in ω.

Theorem 21.29

If $\gamma < \kappa$ and the $\langle C_\alpha \mid \alpha < \gamma \rangle$ are closed unbounded in κ, then $\bigcap_{\alpha<\gamma} C_\alpha$ is closed unbounded in κ.

Proof.

Clearly $\bigcap_{\alpha<\gamma} C_\alpha$ is a closed subset of κ, so it is enough to show that it is unbounded. We argue by induction on γ. If $\gamma = 0$ or $\gamma = 1$ there is nothing to prove. The case of γ a successor ordinal follows from Theorem 21.28, so we may assume that γ is limit. Replacing C_α with $\bigcap_{\beta\leq\alpha} C_\beta$, we may assume that $\alpha < \beta < \gamma \Rightarrow C_\alpha \supseteq C_\beta$. Given a $\nu < \kappa$, construct an increasing sequence $\langle \xi_\alpha \mid \alpha < \gamma \rangle$ with $\nu < \xi_0$ and $\xi_\alpha \in C_\alpha$. Then $\xi = \sup_{\alpha<\gamma} \xi_\alpha \in \kappa$ as κ is regular, and since the C_αs are closed and $\{\xi_\beta \mid \beta \geq \alpha\} \subseteq C_\alpha$, then $\xi \in C_\alpha$ for each $\alpha < \gamma$. □

$\alpha < \kappa$ is closed under $f\colon {}^n\kappa \to \kappa$ if $f(\beta_1, \dots, \beta_n) \in \alpha$ for all $\beta_1, \dots, \beta_n \in \alpha$. The set of all ordinals closed under f is $\mathbf{C}(f)$.

Theorem 21.30

1. $\mathbf{C}(f)$ is closed and unbounded, for all $f\colon {}^n\kappa \to \kappa$.
2. If $C \subseteq \kappa$ is closed and unbounded, then $C \supseteq \mathbf{C}(f)$ for some $f\colon \kappa \to \kappa$.

Proof.

1. As $\alpha < \kappa$ we must find $\gamma \geq \alpha$ which is closed under f. Let

$$\gamma_{i+1} = \sup \{f(\beta_1, \dots, \beta_n) \mid \beta_1, \dots, \beta_n \in \gamma_i\}$$

where $\gamma_0 = \alpha$. By our assumption on κ, we have that $|\{f(\beta_1, \dots, \beta_n) \mid \beta_1, \dots, \beta_n \in \gamma_i\}| \leq |\gamma_i|^n < \kappa$, and hence $\gamma = \sup_i \gamma_i < \kappa$ is the ordinal we are looking for.
Closure of $\mathbf{C}(f)$ in κ is immediate. □

Theorem 21.30

1. $\mathbf{C}(f)$ is closed and unbounded, for all $f\colon {}^{n}\kappa \to \kappa$.
2. If $C \subseteq \kappa$ is closed and unbounded, then $C \supseteq \mathbf{C}(f)$ for some $f\colon \kappa \to \kappa$.

Proof.

2. Let $C \subseteq \kappa$ be a closed unbounded, let g be its enumerating function, and let $f(\alpha) = g(\alpha + 1)$: as $\alpha \leq g(\alpha) < f(\alpha)$, if γ is closed under f, then γ is limit and $C \cap \gamma$ is unbounded in γ. Therefore $\mathbf{C}(f) \subseteq C$. □

Corollary 21.31

If $\mathcal{F}$ is a collection of operations on a regular cardinal κ and $|\mathcal{F}| < \kappa$, then $\bigcap_{f \in \mathcal{F}} \mathbf{C}(f)$, the set of all $\alpha < \kappa$ which are closed under all $f \in \mathcal{F}$, is closed and unbounded in κ.

Therefore if A is an algebraic structure of size κ a regular cardinal with $< \kappa$ many operations and constants (e.g. a group, a ring, a lattice, ...) and $\langle a_\alpha \mid \alpha < \kappa \rangle$ is an enumeration of A, then the set of all $\nu < \kappa$ such that $\{a_\alpha \mid \alpha < \nu\}$ is a substructure of A is closed and unbounded in κ.

Definition 21.32

The **diagonal intersection** of a sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of subsets of κ is $\triangle_{\alpha<\kappa} X_\alpha = \{\beta < \kappa \mid \beta \in \bigcap_{\alpha<\beta} X_\alpha\}$.

If $Y_\alpha = \bigcap_{\beta \leq \alpha} X_\beta$, then $\bigcap_{\alpha<\beta} X_\alpha = \bigcap_{\alpha<\beta} Y_\alpha$ so that $\triangle_{\alpha<\kappa} X_\alpha = \triangle_{\alpha<\kappa} Y_\alpha$.
$\beta \in \bigcap_{\alpha<\beta} X_\alpha$ is equivalent to $\forall \alpha < \beta \, (\beta \in X_\alpha)$, which is equivalent to $\forall \alpha < \kappa \, (\beta \in \alpha \dot{+} 1 \vee \beta \in X_\alpha)$. Therefore

$$\triangle_{\alpha<\kappa} X_\alpha = \bigcap_{\alpha<\kappa} (X_\alpha \cup \alpha \dot{+} 1).$$

If each X_α is closed in κ, then so is $X_\alpha \cup \alpha \dot{+} 1$, and hence $\triangle_{\alpha<\kappa} X_\alpha$ is closed in κ.

Proposition 21.33

If $\kappa > \omega$ and C_α is closed and unbounded in κ for each $\alpha < \kappa$, then $\triangle_{\alpha<\kappa} C_\alpha$ is closed and unbounded in κ.

Proof.

We may assume that $\alpha < \beta \Rightarrow C_\alpha \supseteq C_\beta$. Closure of $C = \triangle_{\alpha<\kappa} C_\alpha$ is immediate so it is enough to check that C is unbounded. Fix $\beta_0 < \kappa$. As $\bigcap_{\nu \leq \gamma} C_\nu$ is unbounded in κ for all $\gamma < \kappa$ (Theorem 21.29), one defines an increasing sequence

$$\beta_0 < \beta_1 < \beta_2 < \cdots < \beta = \sup_n \beta_n$$

such that $\beta_{n+1} \in \bigcap_{\nu \leq \beta_n} C_\nu$. As $n < m \Rightarrow \beta_m \in C_{\beta n}$, the fact that C_{β_n} is closed implies that $\beta = \sup_{m>n} \beta_m \in C_{\beta_n}$, hence $\beta \in \bigcap_n C_{\beta_n} = \bigcap_{\nu<\beta} C_\nu$, that is $\beta_0 < \beta \in C$ as required. □

Definition 21.34

$A \subseteq \kappa$ is **stationary in** κ if $A \cap C \neq \emptyset$ for all closed unbounded $C \subseteq \kappa$.

By Theorem 21.29, a set in $\mathrm{Club}(\kappa)$ is stationary, but not conversely—Exercise 21.58.
A stationary subset of κ is unbounded in κ as it intersects every $(\alpha; \kappa)$.
Thus regularity of κ implies that the stationary sets have size κ.

Theorem 21.35 (Fodor)

Let $S \subseteq \kappa$ be stationary and let $F\colon S \to \kappa$ be such that $\forall \alpha \in S\ (\alpha \neq 0 \Rightarrow F(\alpha) < \alpha)$. Then F is constant on a stationary subset of κ.

Proof.

Towards a contradiction, suppose that $F^{-1}\{\alpha\}$ is non-stationary for all $\alpha < \kappa$, that is

$$\forall \alpha \in \kappa\, \exists C_\alpha \subseteq \kappa\ \left(C_\alpha \text{ closed and unbounded in } \kappa \text{ and } C_\alpha \cap F^{-1}\{\alpha\} = \emptyset\right).$$

By Proposition 21.33, $\triangle_{\alpha<\kappa} C_\alpha$ is closed and unbounded, and since $(0;\kappa)$ is also closed and unbounded, the same is true of $C = (\triangle_{\alpha<\kappa} C_\alpha) \setminus \{0\}$ by Theorem 21.29. Let $\alpha \in S \cap C$: then $\beta \stackrel{\text{def}}{=} F(\alpha) < \alpha$ by definition of F, and $\alpha \in C_\beta$ by definition of diagonal intersection, hence $\alpha \notin F^{-1}\{\beta\}$ that is $F(\alpha) \neq \beta$: a contradiction. $\square$

The exponential function $\kappa \mapsto 2^\kappa$

Rule 1

$\kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda$

Rule 2

$\kappa < \operatorname{cof}(2^\kappa)$, and hence $\kappa^+ \leq 2^\kappa$.

The GCH strengthens Rule 2 by $2^\kappa = \kappa^+$, and therefore $\operatorname{cof}(2^\kappa) = \kappa^+ > \kappa$, for all infinite cardinals κ. By Gödel GCH cannot be refuted from ZFC, and by Cohen CH (and hence GCH) cannot be proved in ZFC.

Easton showed that Rule 1 and Rule 2 are the only restrictions for $\kappa \mapsto 2^\kappa$ when κ is regular.

Example

Each of the following is consistent with ZFC:

- $2^\kappa = \kappa^{++}$ for every regular κ,
- $2^\kappa > \kappa^+$ and that $\forall \lambda < \kappa \left(2^\lambda = \lambda^+\right)$, with κ any regular cardinal.

The situation for *singular cardinals* is much deeper and interesting...
Silver proved that GCH cannot fail first at a singular cardinal of *uncountable cofinality*.

Rule 3

If λ is a limit ordinal of uncountable cofinality and $\{\alpha < \operatorname{cof}(\lambda) \mid 2^{\aleph_\alpha} = \aleph_{\alpha+1}\}$ is stationary in $\operatorname{cof}(\lambda)$, then $2^{\aleph_\lambda} = \aleph_{\lambda+1}$.

In particular, GCH cannot fail first at $\aleph_{\omega_1}$, i.e. if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

The assumption $\omega < \operatorname{cof}(\lambda)$ in in Rule 3 cannot be removed since Magidor proved that GCH can fail first at $\aleph_\omega$: it is consistent that
$\forall n < \omega \left(2^{\aleph_n} = \aleph_{n+1}\right)$ and $2^{\aleph_\omega} > \aleph_{\omega+1}$.
The value $2^{\aleph_\omega}$ cannot be arbitrarily large, as Shelah proved that:

Rule 4

If $\forall n \left(2^{\aleph_n} < \aleph_\omega\right)$, then $2^{\aleph_\omega} < \aleph_{\min(\omega_4,(2^{\aleph_0})^+)}$.

Definition 21.37

A cardinal κ is **strong limit** if $2^\lambda < \kappa$ for all $\lambda < \kappa$. A regular cardinal $\kappa > \omega$ is **weakly inaccessible** if it is limit; it is **strongly inaccessible** if it is strong limit.

If κ is weakly inaccessible then $\kappa = \aleph_\kappa$, but the least fixed point of the $\aleph$ function is of cofinality ω and hence not regular. A strongly inaccessible cardinal is necessarily weakly inaccessible, and GCH guarantees the converse. In the absence of some cardinal arithmetic assumption, the two notions can be distinct; it is possible that $2^{\aleph_0}$ is weakly inaccessible, while if κ is strongly inaccessible then $2^{\aleph_0} < \kappa$.

Lemma 21.38

Assume AC and suppose κ is strongly inaccessible. Then $|\mathrm{V}_\alpha| < \kappa$ for all $\alpha < \kappa$. In particular $|x| < \kappa$ for all $x \in \mathrm{V}_\kappa$.

Proof.

Proceed by induction on α. If $|\mathrm{V}_\alpha| < \kappa$ then $|\mathrm{V}_{\alpha+1}| = 2^{|\mathrm{V}_\alpha|} < \kappa$, as κ is strong limit. If α is limit, then $|\mathrm{V}_\alpha| = |\alpha| \cdot \sup_{\beta<\alpha} |\mathrm{V}_\beta| < \kappa$ by regularity. $\square$

Theorem 21.39

Assume AC. If κ is strongly inaccessible, then $\mathrm{V}_\kappa \vDash \mathsf{ZFC}$.

Proof.

Suppose κ is strongly inaccessible. In order to prove that $\mathrm{V}_\kappa \vDash \mathsf{ZFC}$, by Theorem 19.15 it is enough to show that V_κ satisfies replacement. By part (g) of Theorem 19.22 it is enough to show that if $f\colon a \to \mathrm{V}_\kappa$ with $a \in \mathrm{V}_\kappa$, then there is $b \in \mathrm{V}_\kappa$ such that $\operatorname{ran} f \subseteq b$. Let $g\colon a \to \kappa$, $g(x) =$ the least $\alpha < \kappa$ such that $f(x) \in \mathrm{V}_\alpha$. By Lemma 21.38 $|a| < \kappa$, so $\operatorname{ran} g \subseteq \gamma$ for some $\gamma < \kappa$, and hence $\operatorname{ran} f \subseteq \mathrm{V}_\gamma \in \mathrm{V}_\kappa$. $\square$

The converse of Theorem 21.39 fails: if κ is inaccessible there are many $\alpha < \kappa$ such that $\mathrm{V}_\alpha \vDash \mathsf{ZFC}$ (Theorem 31.22).

Definition 21.40

A **universe** is a transitive set U closed under the operation $x \mapsto \mathscr{P}(x)$, such that $\omega \in U$, and $\forall I \in U \, \forall f \colon I \to U \; \left(\bigcup_{i \in I} f(i) \in U\right)$.

Lemma 21.42

If U is a universe then

1. $x \subseteq y \in U \Rightarrow x \in U$,
2. $x, y \in U \Rightarrow x \cup y \in U$,
3. if $x, y \in U$ then $\{x, y\} \in U$ and hence $(x, y) \in U$,
4. if $x, y \in U$ then $x \times y \in U$ and ${}^{x}y \in U$,
5. if $f \colon I \to U$ and $I \in U$ then $\operatorname{ran} f \in U$ and $f \in U$.

$x \subseteq y \in U \Rightarrow x \in U$

$x \in \mathscr{P}(y) \in U$ so $x \in U$ by transitivity.

$x, y \in U \Rightarrow x \cup y \in U$

$2 \in \omega \in U$, so $2 \in U$ by transitivity. Then $x \cup y = \bigcup_{i \in 2} f(i)$ where $f \colon 2 \to U$ is defined by $f(0) = x$ and $f(1) = y$.

If $x, y \in U$ then $\{x, y\} \in U$ and hence $(x, y) \in U$

If $x \in U$ then $\{x\} \in \mathscr{P}\mathscr{P}(x) \in U$, so $\{x\} \in U$. Thus if $x, y \in U$ then $\{x\}, \{y\} \in U$, so $\{x, y\} \in U$, and therefore $(x, y) \in U$.

if $x, y \in U$ then $x \times y \in U$ and ${}^{x}y \in U$

The result follows from $x \times y \subseteq \mathscr{P}\mathscr{P}(x \cup y)$ and ${}^{x}y \subseteq \mathscr{P}(x \times y)$.

If $f \colon I \to U$ and $I \in U$ then $\operatorname{ran} f \in U$ and $f \in U$

Letting $g \colon I \to U$ be $i \mapsto \{f(i)\}$, then $\operatorname{ran} f = \bigcup_{i \in I} g(i) \in U$. Moreover $f \subseteq I \times \operatorname{ran} f \in U$, whence $f \in U$.

Theorem 21.41 (AC)

U is a universe if and only if $U = \mathrm{V}_\kappa$ for some strongly inaccessible cardinal κ.

Proof

Suppose U is a universe and let $\kappa = U \cap \mathrm{Ord}$.
U is closed under the $\mathbf{S}$ operation, so κ is limit and $\kappa \notin U$. If $\gamma < \kappa$ and $f \colon \gamma \to \kappa$, then $\sup \operatorname{ran} f = \bigcup_{\alpha < \gamma} f(\alpha) \in U$ and hence f cannot be cofinal in κ. It follows that κ is regular. If $2^\lambda \geq \kappa$ for some infinite cardinal $\lambda < \kappa$ there would exist a surjection $f \colon \mathscr{P}(\lambda) \twoheadrightarrow \kappa \subseteq U$. But $\mathscr{P}(\lambda) \in U$ so by Lemma 21.42 $\kappa \in U$, a contradiction. It follows that κ is a strongly inaccessible cardinal.
Let us check that $\mathrm{V}_\alpha \in U$ for all $\alpha < \kappa$, so that $\mathrm{V}_\kappa \subseteq U$. As U is closed under the $\mathscr{P}$ operation, then $\bar{\kappa} = \{\alpha < \kappa \mid \mathrm{V}_\alpha \in U\}$ is a limit ordinal: if $\bar{\kappa} < \kappa$ then using the function $\bar{\kappa} \to U$, $\alpha \mapsto \mathrm{V}_\alpha$, we would have that $\mathrm{V}_{\bar{\kappa}} = \bigcup_{\alpha < \bar{\kappa}} \mathrm{V}_\alpha \in U$, so that $\bar{\kappa} \in \bar{\kappa}$, a contradiction. Therefore $\mathrm{V}_\kappa \subseteq U$.

(continues)

Proof (continues).

Recall that $\kappa = U \cap \mathrm{Ord}$ is inaccessible, and $\mathrm{V}_\kappa \subseteq U$.
We prove that $U \subseteq \mathrm{V}_\kappa$. Towards a contradiction, let $x \in U \setminus \mathrm{V}_\kappa$ be of least rank: then $\mathrm{rank}(x) \geq \kappa$ so that the map $x \to \kappa$, $y \mapsto \mathrm{rank}(y)$, is cofinal so that $\kappa = \sup_{y \in x} \mathrm{rank}(y) \in U$, a contradiction. Therefore $\mathrm{V}_\kappa = U$.

Suppose now κ is a strongly inaccessible cardinal, and let us check that V_κ is a universe. Suppose $f \colon I \to \mathrm{V}_\kappa$ with $I \in \mathrm{V}_\kappa$. Then the function $I \to \kappa$, $i \mapsto \mathrm{rank}(f(i))$, is bounded in κ, since $|I| < \kappa$, so $\operatorname{ran} f \subseteq \mathrm{V}_\alpha$ for some $\alpha < \kappa$. Therefore $\bigcup_{i \in I} f(i) \subseteq \mathrm{V}_\alpha$, and hence $\bigcup_{i \in I} f(i) \in \mathrm{V}_{\alpha+1} \subseteq \mathrm{V}_\kappa$.
The other clauses in the definition of universe are immediate. □