7. Show that there exists a unique function $g \in L(3p_0)$ such that the Laurent series for g (in a coordinate z centered at p_0) has the form
$$g(z) = \frac{1}{z^3} + b_1 z^1 + \dots.$$
8. Show that $g(x) = -g(-x)$ for all x in X. (This is equivalent to the Laurent series for g (in a coordinate z centered at p_0) having only odd degree terms.) Hence the Laurent series for g actually has the form
$$g(z) = \frac{1}{z^3} + b_1 z^1 + b_3 z^3 + \dots.$$
9. Find the Laurent series for f^2, f^3, and g^2 (up through the 'z' term) in terms of the above-written Laurent series for f and g.
10. Show that $g^2 = f^3 + Af + B$ for some constants A and B.
11. Show that the polynomial $w^3 + Aw + B$ has no double roots. (Hint: suppose that α is a double root. Show that the meromorphic function $g/(f - \alpha)$ is a square root of a function in $L(2p_0)$.)

G. Show that given any two meromorphic functions f and g on X, there is a divisor D such that f and g are both in $L(D)$.

H. Suppose that X is a compact Riemann surface and $D > 0$ is a strictly positive divisor on X such that $\dim L(D) = 1 + \deg(D)$. Conclude that there exists a point $p \in X$ such that $\dim L(p) = 2$. Conclude that X is isomorphic to the Riemann Sphere.

I. Let X be a Riemann surface, and let E be any divisor on X. Suppose that D is a nonnegative divisor with finite support. Show that $L(E) \subseteq L(E + D)$ has codimension at most $\deg(D)$.

4. Divisors and Maps to Projective Space

One of the primary ways of understanding Riemann surfaces is to map them into a projective space. If we can exhibit a Riemann surface X as holomorphically embedded in a projective space, that is, as a smooth projective curve, the tools of algebraic geometry can come into play, in particular the use of hyperplane divisors, etc. Therefore, via intersections, embeddings of X into projective space give rise to divisors; the converse is also true, as we will see.

Holomorphic Maps to Projective Space. The first task is to understand what it means to be a holomorphic map to $\mathbb{P}^n$. The condition is local on the and we must now check that Φ is an isomorphism.

The inverse Ψ to Φ is readily defined. Fix an open set U, and a section $r \in \mathcal{O}(U)$. Define $\Psi(r)$ to be the collection (s_i), where $s_i = r|_{U \cap U_i} \cdot f_i \otimes \phi_i$. This defines a sheaf map, and we leave it to the reader to check that Φ and Ψ are inverses of one another.

This proves the following.

PROPOSITION 1.14. *Let X be an algebraic curve. Then the set $\mathrm{Inv}(X)$ of isomorphism classes of invertible sheaves on X forms an abelian group whose operation is induced by the tensor product. The identity is the class of the sheaf $\mathcal{O}$ of regular functions on X. The inverse of the class of an invertible sheaf is the class of the inverse invertible sheaf.*

We will see in Section 3 that the map sending a divisor D to the invertible sheaf $\mathcal{O}[D]$ induces an isomorphism from the Picard group $\mathrm{Pic}(X)$ of divisors modulo linear equivalence, to the group $\mathrm{Inv}(X)$ of invertible sheaves. For this reason all algebraic geometers refer to $\mathrm{Inv}(X)$ as the Picard group of X.

Problems XI.1

A. Show that the kernel of a sheaf map of $\mathcal{O}$-modules is an $\mathcal{O}$-module.

B. Show that $\mathcal{O}_{X,alg}[D]$, $\Omega^1_{X,alg}[D]$, and $\mathcal{M}_{X,alg}$ are sheaves of $\mathcal{O}$-modules.

C. Show that if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then for every point $p \in X$, the stalk $\mathcal{F}_p$ is a module over the stalk $\mathcal{O}_p$.

D. Prove or disprove: a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is invertible if and only if for every point $p \in X$, the stalk $\mathcal{F}_p$ is a free rank one $\mathcal{O}_p$-module.

E. Show that the presheaf defined by (1.7) may not satisfy the sheaf axiom, by considering the Riemann Sphere X, a point $p \in X$ (say $p = \infty$), and the two invertible sheaves $\mathcal{O}_{X,alg}[-p]$ and $\mathcal{O}_{X,alg}[p]$. Specifically, show that there is an open cover of X and sections of the presheaf over each open set in the cover, which agree on the intersections, but which do not come from a global section of the presheaf.

F. Show that if $\{U_i\}$ is any open covering of X such that both $\mathcal{F}$ and $\mathcal{G}$ are trivialized over each U_i, and we define a sheaf $\mathcal{T}$ on X by setting

$$\begin{aligned} \mathcal{T}(U) &= \{(s_i) \in \prod_i \mathcal{F}(U \cap U_i) \otimes_{\mathcal{O}(U \cap U_i)} \mathcal{G}(U \cap U_i) \mid \\ &\qquad s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j} \text{ for all } i, j\}, \end{aligned}$$

then the sheaf $\mathcal{T}$ is isomorphic to the tensor product sheaf $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.

G. Prove Lemma 1.10.

H. Show that the presheaf defined by (1.11) may not satisfy the sheaf axiom, by considering the Riemann Sphere X, a point $p \in X$ (say $p = \infty$), and the invertible sheaf $\mathcal{O}_{X,alg}[-p]$. Specifically, show that there is an open cover of X and sections of the inverse presheaf over each open set in the cover, which agree on the intersections, but which do not come from a global section of the inverse presheaf.

Since as noted above the function g does not change when we change line bundle charts, this formula in fact holds for all p in X. Hence $\text{div}(s_2) = \text{div}(g) + \text{div}(s_1)$, proving that $\text{div}(s_2)$ is linearly equivalent to $\text{div}(s_1)$. □

We therefore obtain a function from the set $\text{LB}(X)$ of isomorphism classes of line bundles on X to the Picard group $\text{Pic}(X)$ of divisors modulo linear equivalence, by sending the class of a line bundle L to the divisor of any of its rational sections. We will see in the next section that this is a bijection.

Problems XI.2

A. Check that the additions and scalar multiplications induced on the fiber L_p of a line bundle L over a point p by two different compatible line bundle charts are in fact the same.

B. Check that the functions ϕ_i used to define the tautological line bundle for a map to projective space are indeed line bundle charts.

C. As a projective space, the algebraic curve $\mathbb{P}^1$ has a tautological line bundle L. Show that L may be defined by an atlas with two line bundle charts, supported over the two standard charts of $\mathbb{P}^1$. Find the transition functions for L.

D. Show that the composition of two line bundle homomorphisms is a line bundle homomorphism.

E. Check that the line bundle L constructed in the proof of Proposition 2.9 is unique.

F. Show that the tautological line bundle on $\mathbb{P}^1$ is one of the line bundles L_n defined in Example 2.10. Which n is it?

G. Suppose that L is a line bundle on an algebraic curve X, U is an open subset of X, s is a function from U to L satisfying $\pi \circ s = \text{id}_U$, and suppose that s satisfies condition (ii) of Definition 2.13 for all line bundle charts in some line bundle atlas for L. Show that s is a regular section of L over U, i.e., that s satisfies the condition (ii) for all line bundle charts for L.

H. Let L be a line bundle on an algebraic curve X. Show that a line bundle homomorphism $\alpha : \mathbb{C} \times X \to L$ from the trivial line bundle to L induces a global regular section s_α of L, by setting $s_\alpha(p) = \alpha(1, p)$ for each $p \in X$. Show that every global regular section of L is obtained from a unique such line bundle homomorphism α.

I. Show that if L is the trivial line bundle on X, then the invertible sheaf $\mathcal{O}\{L\}$ is isomorphic to the sheaf $\mathcal{O}$ of regular functions on X.

J. Let $\alpha : L_1 \to L_2$ be a line bundle homomorphism. Show that α induces a sheaf map from $\mathcal{O}\{L_1\}$ to $\mathcal{O}\{L_2\}$ by sending a regular section s of L over U to the composition $\alpha \circ s$.

K. Show that if $\mathbb{K}$ is the canonical bundle on X, then the sheaf $\mathcal{O}\{\mathbb{K}\}$ of regular sections of $\mathbb{K}$ is isomorphic to the sheaf of regular 1-forms Ω^1.

L. Prove Lemma 2.21.

M. Define the degree of a line bundle L on an algebraic curve to be the degree