We will be able to give a proof of the above statement for an arbitrary compact Riemann surface shortly, along slightly different lines.

Problems II.3

A. Verify Example 3.4: if Y is the complex plane $\mathbb{C}$, prove that a holomorphic map $F : X \to Y$ is simply a holomorphic function on X.

B. Prove all the statements of Lemma 3.5.

C. Show that under the isomorphism between $\mathbb{P}^1$ and the Riemann Sphere $\mathbb{C}_\infty$, the points $[z : 1]$ are sent to the finite points z, and the point $[1 : 0]$ is sent to ∞.

D. Explicitly write down the inverse holomorphic map to the isomorphism from $\mathbb{P}^1$ to $\mathbb{C}_\infty$ given in the proof of Lemma 3.7. Check everything necessary.

E. Let $\pi : \mathbb{C} \to X = \mathbb{C}/L$ be the natural projection map defining a complex torus X. Let Y be a Riemann surface. Show that a map $F : X \to Y$ is holomorphic if and only if $F \circ \pi : \mathbb{C} \to Y$ is holomorphic. Deduce that the projection map π is a holomorphic map.

le fibre di π non sono finite

F. Let $f(z, w)$ and $g(z, w)$ be homogeneous polynomials of the same degree with no common factor, and not both identically zero. Show that the map $F : \mathbb{P}^1 \to \mathbb{P}^1$ defined by sending $[z : w]$ to $[f(z, w) : g(z, w)]$ is well defined and holomorphic. What if f and g have a common factor?

G. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible 2-by-2 matrix over $\mathbb{C}$. Show that the map $F_A : \mathbb{P}^1 \to \mathbb{P}^1$ sending $[z : w]$ to $[az + bw : cz + dw]$ is an automorphism of $\mathbb{P}^1$. For which matrices A is F_A the identity? Show that $F_{AB} = F_A \circ F_B$.

H. Show that after identifying $\mathbb{P}^1$ with $\mathbb{C}_\infty$, the automorphism F_A defined above takes $z \in \mathbb{C}_\infty$ to $(az + b)/(cz + d)$; hence it is a linear fractional transformation.

I. Let X be a compact Riemann surface and f a nonconstant meromorphic function on X. Show that f must have a zero on X, and must have a pole on X.

J. Prove that, given a meromorphic function f on a Riemann surface X, the associated map $F : X \to \mathbb{C}_\infty$ is holomorphic. Verify the 1-1 correspondence of Proposition 3.13.

K. Recall that a *lattice* $L \subset \mathbb{C}$ is an additive subgroup generated (over $\mathbb{Z}$) by two complex numbers ω_1 and ω_2 which are linearly independent over $\mathbb{R}$. Thus $L = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$.
   1. Suppose that $L \subseteq L'$ are two lattices in $\mathbb{C}$. Show that the natural map from $\mathbb{C}/L$ to $\mathbb{C}/L'$ is holomorphic, and is biholomorphic if and only if $L = L'$.
   2. Let L be a lattice in $\mathbb{C}$ and let α be a nonzero complex number. Show that αL is a lattice in $\mathbb{C}$ and that the map

$$\phi : \mathbb{C}/L \to \mathbb{C}/(\alpha L)$$

sending the coset $z + L$ to $(\alpha z) + (\alpha L)$ is a well defined biholomorphic map.

3. Show that every torus $\mathbb{C}/L$ is isomorphic to a torus which has the form $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where τ is a complex number with strictly positive imaginary part.

4. Global Properties of Holomorphic Maps

Local Normal Form and Multiplicity. It may seem strange to have the first part of a section on global properties dealing with a completely local concept. However, most global properties actually state that some function of local invariants is constant. This is the case in our situation, and so we must introduce the local invariant before proceeding.

A holomorphic map between two Riemann surfaces has a standard normal form in some local coordinates: essentially, every map looks like a power map. This we now present.

PROPOSITION 4.1 (LOCAL NORMAL FORM). *Let $F : X \to Y$ be a holomorphic map defined at $p \in X$, which is not constant. Then there is a unique integer $m \geq 1$ which satisfies the following property: for every chart $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$, there exists a chart $\phi_1 : U_1 \to V_1$ on X centered at p such that $\phi_2(F(\phi_1^{-1}(z))) = z^m$.*

PROOF. Fix a chart ϕ_2 on Y centered at $F(p)$, and choose any chart $\psi : U \to V$ on X centered at p. Then the Taylor series for the function $T(w) = \phi_2(F(\psi^{-1}(w)))$ must be of the form

$$T(w) = \sum_{i=m}^{\infty} c_i w^i$$

with $c_m \neq 0$, and $m \geq 1$ since $T(0) = 0$. Thus we have $T(w) = w^m S(w)$ where $S(w)$ is a holomorphic function at $w = 0$, and $S(0) \neq 0$. In this case there exists a function $R(w)$ holomorphic near 0 such that $R(w)^m = S(w)$, so that $T(w) = (wR(w))^m$. Let $\eta(w) = wR(w)$; since $\eta'(0) \neq 0$, we see that near 0 the function η is invertible (by the Implicit Function Theorem), and of course holomorphic. Hence the composition $\phi_1 = \eta \circ \psi$ is also a chart on X defined and centered near p. If we think of η as defining a new coordinate z (via $z = \eta(w)$), we see that z and w are related by $z = wR(w)$. Thus

$$\begin{aligned}
\phi_2(F(\phi_1^{-1}(z))) &= \phi_2(F(\psi^{-1}(\eta^{-1}(z)))) \\
&= T(\eta^{-1}(z)) \\
&= T(w) \\
&= (wR(w))^m \\
&= z^m.
\end{aligned}$$

Therefore

$$\begin{aligned}
2g(X)-2 &= -e(X) \\
&= -v' + e' - t' \\
&= -\deg(F)v + \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1] + \deg(F)e - \deg(F)t \\
&= -\deg(F)e(Y) + \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1] \\
&= \deg(F)(2g(Y)-2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1],
\end{aligned}$$

the last equality holding because every ramification point of F is a vertex of X. $\square$

We may view this proof as resolving two different ways of computing preimages. If we "count properly", we take into account the ramification of the map and all of the multiplicities. If we count "naively", we get a computation of the Euler number. Putting these two things together gives Hurwitz's formula.

Problems II.4

A. Verify the statement in Example 4.3 that chart maps have constant multiplicity one. Is the converse true? (I.e., is every holomorphic map from an open set in X to an open set in $\mathbb{C}$ with constant multiplicity one, a chart map?) No.

B. Let F be a holomorphic map between Riemann surfaces. Prove that the set of points p with $\mathrm{mult}_p(F) \geq 2$ forms a discrete subset of the domain by using the Local Normal Form.

C. Let $F: X \to Y$ and $G: Y \to Z$ be two nonconstant holomorphic maps between Riemann surfaces. Show that if $p \in X$, then $\mathrm{mult}_p(G \circ F) = \mathrm{mult}_p(F)\,\mathrm{mult}_{F(p)}(G)$. Show that if f is a meromorphic function on Y, then $\mathrm{ord}_p(f \circ F) = \mathrm{mult}_p(F)\,\mathrm{ord}_{F(p)}(f)$.

D. Explicitly triangulate the sphere, the disk, and the cylinder and verify that they have Euler numbers 2, 1, and 0 respectively.

E. Show that if f is a holomorphic function at p, and $\mathrm{mult}_p(f) = 1$ (considering f as a holomorphic map locally to $\mathbb{C}$), then f is a local coordinate function at p.

F. Let f be a global meromorphic function on a compact Riemann surface X. Show that f is a local coordinate at all but finitely many points of X.

G. Let $f(z) = z^3/(1-z^2)$, considered as a meromorphic function on the Riemann Sphere $\mathbb{C}_\infty$. Find all points p such that $\mathrm{ord}_p(f) \neq 0$. Consider the associated map $F: \mathbb{C}_\infty \to \mathbb{C}_\infty$. Show that F has degree 3 as a holomorphic map, and find all of its ramification and branch points. Verify Hurwitz's formula for this map F.

H. Let $f(z) = 4z^2(z-1)^2/(2z-1)^2$, considered as a meromorphic function on the Riemann Sphere $\mathbb{C}_\infty$. Find all points p such that $\mathrm{ord}_p(f) \neq 0$. Consider the associated map $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Show that F has degree 4 as a holomorphic map, and find all of its ramification and branch points. Verify Hurwitz's formula for this map F.

I. Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces.

1. Show that if $Y \cong \mathbb{P}^1$, and F has degree at least two, then F must be ramified.
2. Show that if X and Y both have genus one, then F is unramified.
3. Show that $g(Y) \leq g(X)$ always.
4. Show that if $g(Y) = g(X) \geq 2$, then F is an isomorphism.

J. Let X be the projective plane curve of degree d defined by the homogeneous polynomial $F(x, y, z) = x^d + y^d + z^d$. This curve is called the *Fermat curve of degree d*. Let $\pi : X \to \mathbb{P}^1$ be given by $\pi[x : y : z] = [x : y]$.

1. Check that the Fermat curve is smooth.
2. Show that π is a well defined holomorphic map of degree d.
3. Find all ramification and branch points of π.
4. Use Hurwitz's formula to compute the genus of the Fermat curve: you should get

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

K. Let U be the affine plane curve defined by $x^2 = 3 + 10t^4 + 3t^8$. Let V be the affine plane curve defined by $w^2 = z^6 - 1$. Show that both curves are smooth. Show that the function $F : U \to V$ defined by $z = (1+t^2)/(1-t^2)$ and $w = 2tx/(1-t^2)^3$ is holomorphic and nowhere ramified whenever $t \neq \pm 1$.

Further Reading

The basic material on singularities of complex functions is standard fare in all texts on complex variables; each of the texts mentioned at the end of Chapter I have plenty on this, and also sections on harmonic functions, which are sometimes given short shrift in a first course.

Many authors introduce meromorphic functions on a torus (also known as *elliptic functions*) via the Weierstrass P-function; this is the approach taken for example in [**Ahlfors66**], [**JS87**], [**Lang85**], and [**Lang87**]. We have taken the approach of theta-functions, to emphasize the analogy between ratios of theta-functions (on a torus) and ratios of homogeneous polynomials (on the projective line); this is also the approach of [**Clemens80**]. For (much) more depth on theta-functions, see [**R-F74**], [**Gunning76**], and [**Mumford83**].

We have mentioned Shafarevich's text [**Shafarevich77**] for the Nullstellensatz; there are many other references, many in texts in algebra, for example, [**Z-S60**], [**AM69**], [**Hungerford74**], and [**Lang84**]; students just starting out may find the treatment in [**Artin91**] less steep. The Nullstellensatz is at the