16

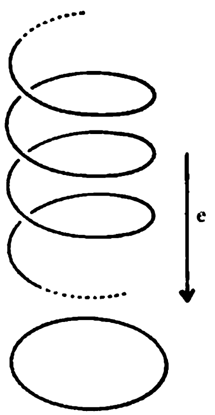
The fundamental group of a circle

Except for some trivial cases we have not, so far, calculated the fundamental group of a space. In this chapter we shall calculate the fundamental group of the circle S^1, the answer being $\mathbb{Z}$ the integers. Intuitively we see this result as follows. A closed path f in S^1 based at $1 \in S^1$ winds a certain number of times around the circle; this number is called the winding number or degree of f. (Start with $f(0) = 1$ and consider $f(t)$ as t increases; every time we go once around the circle in an anticlockwise direction record a score of +1, every time we go once around in a clockwise direction score −1. The total score is the winding number or degree of f.) Thus to each closed path f based at 1 we get an integer. It turns out that two closed paths are equivalent (i.e. homotopic rel $\{0,1\}$) if and only if their degrees agree. Finally, for each integer n there is a closed path of degree n.

To get a more precise definition of the degree of a closed path we consider the real numbers $\mathbb{R}$ mapping onto S^1 as follows.

$$e: \mathbb{R} \to S^1,$$
$$t \to \exp(2\pi i t).$$

Figure 16.1



Geometrically we think of the reals as a spiral with e being the projection mapping (see Figure 16.1). Note that $e^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$. The idea now is that if we are given $f\colon I \to S^1$ with $f(0) = f(1) = 1$ then we show that there is a unique map $\tilde{f}\colon I \to \mathbb{R}$ with $\tilde{f}(0) = 0$ and $e\tilde{f} = f$ (the map $\tilde{f}$ is called a *lift* of f) Since $f(1) = 1$ we must have $\tilde{f}(1) \in e^{-1}(1) = \mathbb{Z}$; this integer is defined to be the degree of f. We then go on to show that if f_0 and f_1 are equivalent paths in S^1 then $\tilde{f}_0(1) = \tilde{f}_1(1)$. This leads to a function $\pi(S^1,1) \to \mathbb{Z}$ which we finally show is an isomorphism of groups.

The 'method of calculation' of $\pi(S^1,1)$ that we shall be presenting generalizes to some other spaces; see the subsequent three chapters. In fact the next lemma is the starting point for a crucial definition in Chapter 17.

16.1 Lemma

Let U be any open subset of $S^1 - \{1\}$ and let $V = I \cap e^{-1}(U) \subseteq \mathbb{R}$. Then $e^{-1}(U)$ is the disjoint union of the open sets $V + n = \{v+n; v \in V\}$, $n \in \mathbb{Z}$, each of which is mapped homeomorphically onto U by e.

Proof We assume that U is an open interval, i.e.

$$U = \{\exp(2\pi i t); 0 \leq a < t < b \leq 1\}$$

for some a,b. Then $V = (a,b)$ and $V + n = (a+n, b+n)$. It is clear that $e^{-1}(U)$ is the disjoint union of the open sets $V + n$ ($n \in \mathbb{Z}$). Let e_n denote the restriction of e to $(a+n,b+n)$. Clearly e_n is continuous and bijective. To check that e_n^{-1} is continuous we consider $(a+n, b+n)$ and let $W \subseteq (a+n, b+n)$ be a closed (and hence compact) subset. Since W is compact and S^1 is Hausdorff, e_n induces a homeomorphism $W \to e_n(W)$ by Theorem 8.8. In particular $e_n(W)$ is compact and hence closed. This shows that if W is a closed subset then $e_n(W)$ is also closed; thus e_n^{-1} is continuous and hence e_n is a homeomorphism.

16.2 Exercise

Show that the above holds for $S^1 - \{x\}$, where x is any point of S^1.

16.3 Corollary

If $f\colon X \to S^1$ is not surjective then f is null homotopic.

Proof If $x \notin$ image (f) then $S^1 - \{x\}$ is homeomorphic to (0,1) which is contractible. ($x = \exp(2\pi i s)$ for some s and $S^1 = \{\exp(2\pi i t); s \leq t < 1+s\}$.)

We come now to the first major result of this chapter: the so-called *path* *lifting theorem* (for e: $\mathbb{R} \to S^1$).

16.4 Theorem

Any continuous map $f: I \to S^1$ has a lift $\tilde{f}: I \to \mathbb{R}$. Furthermore given $x_0 \in \mathbb{R}$ with $e(x_0) = f(0)$ there is a unique lift $\tilde{f}$ with $\tilde{f}(0) = x_0$.

Proof For each $x \in S^1$ let U_x be an open neighbourhood of x such that $e^{-1}(U_x)$ is the disjoint union of open subsets of $\mathbb{R}$ each of which are mapped homeomorphically onto U_x by e. The set $\{ f^{-1}(U_x);\ x \in S^1 \}$ may be expressed in the form $\{ (x_j, y_j) \cap I;\ j \in J \}$ which is an open cover of I. Since I is compact there is a finite subcover of the form

$$[0, t_1 + \epsilon_1), (t_2 - \epsilon_2, t_2 + \epsilon_2), \ldots, (t_n - \epsilon_n, 1]$$

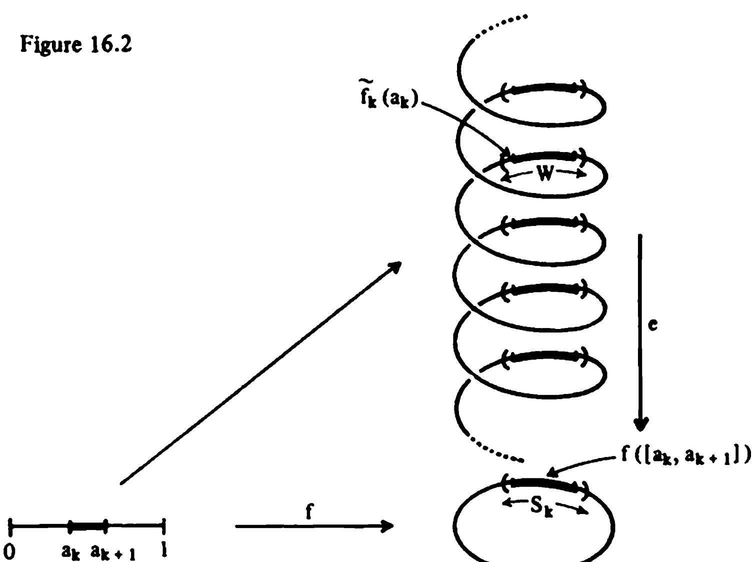
with $t_i + \epsilon_i > t_{i+1} - \epsilon_{i+1}$ for $i = 1, 2, \ldots, n-1$. Now choose $a_i \in (t_{i+1} - \epsilon_{i+1}, t_i + \epsilon_i)$ for $i = 1, 2, \ldots, n-1$ so that

$$0 = a_0 < a_1 < a_2 < \ldots < a_n = 1.$$

Obviously $f([a_i, a_{i+1}]) \subset S^1$, but more so $f([a_i, a_{i+1}])$ is contained in an open subset S_i of S^1 such that $e^{-1}(S_i)$ is the disjoint union of open subsets of $\mathbb{R}$ each of which are mapped homeomorphically onto S_i by e.

We shall define liftings $\tilde{f}_k$ inductively over $[0, a_k]$ for $k = 0, 1, \ldots, n$ such that $\tilde{f}_k(0) = x_0$. For $k = 0$ this is trivial: $\tilde{f}_0(0) = x_0$; we have no choice.

Figure 16.2



Suppose that $\tilde{f}_k$: $[0,a_k] \to \mathbb{R}$ is defined and is unique. Recall that $f([a_k,a_{k+1}]) \subseteq S_k$ and that $e^{-1}(S_k)$ is the disjoint union of $\{W_j; j \in J\}$ with $e|W_j$: $W_j \to S_k$ being a homeomorphism for each $j \in J$. Now $\tilde{f}_k(a_k) \in W$ for some unique member W of $\{W_j; j \in J\}$; see Figure 16.2. Any extension $\tilde{f}_{k+1}$ must map $[a_k,a_{k+1}]$ into W since $[a_k,a_{k+1}]$ is path connected. Since the restriction $e|W$: $W \to S_k$ is a homeomorphism there is a unique map ρ: $[a_k,a_{k+1}] \to W$ such that $e\rho = f|[a_k,a_{k+1}]$ (in fact $\rho = (e|W)^{-1} f$). Now define $\tilde{f}_{k+1}$ by

$$\tilde{f}_{k+1}(s) = \begin{cases} \tilde{f}_k(s) & 0 \leq s \leq a_k, \\ \rho(s) & a_k \leq s \leq a_{k+1}, \end{cases}$$

which is continuous by the glueing lemma since $\tilde{f}_k(a_k) = \rho(a_k)$ and is unique by construction. By induction we obtain $\tilde{f}$.

Using this theorem we can define the degree of a closed path in S^1. Let f be a closed path in S^1 based at 1 and let $\tilde{f}$: $I \to \mathbb{R}$ be the unique lift with $\tilde{f}(0) = 0$. Since $e^{-1}(f(1)) = e^{-1}(1) = \mathbb{Z}$ we see that $\tilde{f}(1)$ is an integer which is defined to be the *degree* of f. To show that equivalent paths have the same degree we shall first show that equivalent paths have equivalent lifts. To do this we replace I by I^2 in the previous theorem to obtain.

16.5 Lemma

Any continuous map F: $I^2 \to S^1$ has a lift $\tilde{F}$: $I^2 \to \mathbb{R}$. Furthermore given $x_0 \in \mathbb{R}$ with $e(x_0) = F(0,0)$ there is a unique lift $\tilde{F}$ with $\tilde{F}(0,0) = x_0$.

Proof The proof is quite similar to that of Theorem 16.4. Since I^2 is compact we find

$$0 = a_0 < a_1 < \ldots < a_n = 1,$$
$$0 = b_0 < b_1 < \ldots < b_m = 1,$$

such that $F(R_{i,j}) \subset S^1$, where $R_{i,j}$ is the rectangle

$$R_{i,j} = \{ (t,s) \in I^2 ; a_i \leq t \leq a_{i+1}, b_j \leq s \leq b_{j+1} \}.$$

The lifting $\tilde{F}$ is defined inductively over the rectangles

$$R_{0,0}, R_{0,1}, \ldots, R_{0,m}, R_{1,0}, R_{1,1}, \ldots$$

by a process similar to that in Theorem 16.4. We leave the details for the reader.

As a corollary we have the so-called *monodromy theorem* for e: $\mathbb{R} \to S^1$, which tells us that equivalent paths have the same degree.

16.6 Corollary

Suppose that f_0 and f_1 are equivalent paths in S^1 based at 1. If $\tilde{f}_0$ and $\tilde{f}_1$ are lifts with $\tilde{f}_0(0) = \tilde{f}_1(0)$ then $\tilde{f}_0(1) = \tilde{f}_1(1)$.

Proof Let F be the homotopy rel $\{0,1\}$ between f_0 and f_1. It lifts uniquely to $\tilde{F}: I^2 \to R$ with $\tilde{F}(0,0) = \tilde{f}_0(0) = \tilde{f}_1(0)$. Since $F(t,0) = f_0(t)$ and $F(t,1) = f_1(t)$, we have $\tilde{F}(t,0) = \tilde{f}_0(t)$ and $\tilde{F}(t,1) = \tilde{f}_1(t)$. Also, $\tilde{F}(1,t)$ is a path from $\tilde{f}_0(1)$ to $\tilde{f}_1(1)$ since $F(1,t) = f_0(1) = f_1(1)$. But $\tilde{F}(1,t) \in e^{-1}(f_0(1)) \cong \mathbb{Z}$, which means that $\tilde{F}(1,t)$ is constant and hence $\tilde{f}_0(1) = \tilde{f}_1(1)$ thus completing the proof. Note that in fact $\tilde{F}$ provides a homotopy rel $\{0,1\}$ between $\tilde{f}_0$ and $\tilde{f}_1$.

We are now in a position to calculate the fundamental group of the circle.

16.7 Theorem

$\pi(S^1,1) \cong \mathbb{Z}$.

Proof Define $\varphi: \pi(S^1,1) \to \mathbb{Z}$ by $\varphi([f]) = \deg(f)$, the degree of f. Recall that $\deg(f) = \tilde{f}(1)$ where $\tilde{f}$ is the unique lift of f with $\tilde{f}(0) = 0$. The function φ is well defined by Corollary 16.6. We shall show that φ is an isomorphism of groups.

First we show that φ is a homomorphism. Let $\ell_a(f)$ denote the lift of f beginning at $a \in e^{-1}(f(0))$. Thus $\ell_0(f) = \tilde{f}$ and $\ell_a(f)(t) = \tilde{f}(t) + a$ for a path in S^1 beginning at 1. It is clear that

$$\ell_a(f * g) = \ell_a(f) * \ell_b(g)$$

where $b = \tilde{f}(1) + a$. Thus if $[f], [g] \in \pi(S^1,1)$ then

$$\begin{aligned}
\varphi([f]\,[g]) &= \varphi([f * g] = \widetilde{f * g}(1) \\
&= \ell_0(f * g)(1) \\
&= (\ell_0(f) * \ell_b(g))(1) \quad \text{where } b = \tilde{f}(1) \\
&= \ell_b(g)(1) \\
&= b + \tilde{g}(1) \\
&= \tilde{f}(1) + \tilde{g}(1) \\
&= \varphi([f]) + \varphi([g])
\end{aligned}$$

which shows that φ is a homomorphism.

To show that φ is surjective is rather easy: given $n \in \mathbb{Z}$ let $g: I \to R$ be given by $g(t) = nt$; then $eg: I \to S^1$ is a closed path based at 1. Since g is the lift of eg with $g(0) = 0$ we have $\varphi([eg]) = \deg(eg) = g(1) = n$ which shows that φ is surjective.

To show that φ is injective we suppose that $\varphi([f]) = 0$, i.e. $\deg(f) = 0$.

This means that the lift $\tilde{f}$ of f satisfies $\tilde{f}(0) = \tilde{f}(1) = 0$. Since $\mathbb{R}$ is contractible we have $\tilde{f} \simeq \epsilon_0$ (rel $\{0,1\}$); in other words there is a map F: $I^2 \to \mathbb{R}$ with $F(0,t) = \tilde{f}(t)$, $F(1,t) = 0$ and $F(t,0) = F(t,1) = 0$. Indeed $F(s,t) = (1-s)\tilde{f}(t)$. But eF: $I^2 \to S^1$ with $eF(0,t) = f(t)$, $eF(1,t) = 1$, $eF(t,0) = eF(t,1) = 1$ and so $f \simeq \epsilon_1$ (rel $\{0,1\}$), i.e. $[f] = 1 \in \pi(S^1, 1)$, which proves that φ is injective and hence φ is an isomorphism.

This completes the proof of the main result of this chapter. As a corollary we immediately obtain:

16.8 Corollary

The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

We close the chapter by giving two applications. The first is well known and is the *fundamental theorem of algebra.*

16.9 Corollary

Every non-constant complex polynomial has a root.

Proof We may assume without loss of generality that our polynomial has the form

$$p(z) = a_0 + a_1 z + \dots + a_{k-1} z^{k-1} + z^k$$

with $k \geq 1$. Assume that p has no zero (i.e. no root). Define a function G: $I \times [0,\infty) \to S^1 \subset \mathbb{C}$ by

$$G(t,r) = \frac{p(r\exp(2\pi it))}{|p(r\exp(2\pi it))|} \quad \frac{|p(r)|}{p(r)}$$

for $0 \leq t \leq 1$ and $r \geq 0$. Clearly G is continuous. Define F: $I^2 \to S^1$ by

$$F(t,s) = \begin{cases} G(t, s/(1-s)) & 0 \leq t \leq 1, 0 \leq s < 1, \\ \exp(2\pi ikt) & 0 \leq t \leq 1, s = 1. \end{cases}$$

By observing that

$$\lim_{s\to 1} F(t,s) = \lim_{s\to 1} G(t, s/(1-s)) = \lim_{r\to\infty} G(t,r) = (\exp(2\pi it))^k$$

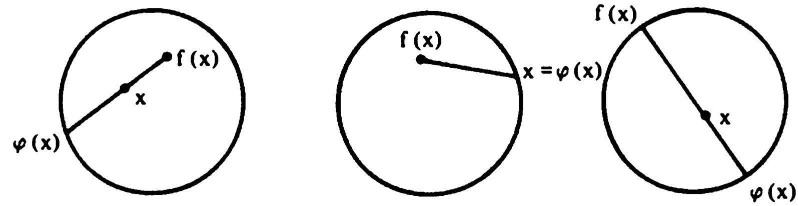
we see that F is continuous. Also, we see that F is a homotopy rel $\{0,1\}$ between $f_0(t) = F(t,0)$ and $f_1(t) = F(t,1)$. But $f_0(t) = 1$ and $f_1(t) = \exp(2\pi ikt)$, so that $\deg(f_0) = 0$ while $\deg(f_1) = k$, which is a contradiction (unless $k = 0$).

The second application comes under the title of *Brouwer's fixed point theorem in the plane.* Recall that in Chapter 10 we proved a fixed point theorem for I; the next result is the analogous theorem for D^2. The result is also true in higher dimensions but the proof requires tools other than the fundamental group.

16.10 Corollary

Any continuous map $f: D^2 \to D^2$ has a fixed point, i.e. a point x such that $f(x) = x$.

Figure 16.3



Proof Suppose to the contrary that $x \neq f(x)$ for all $x \in D^2$. Then we may define a function $\varphi: D^2 \to S^1$ by setting $\varphi(x)$ to be the point on S^1 obtained from the intersection of the line segment from f(x) to x extended to meet S^1; see Figure 16.3. That φ is continuous is obvious. Let $i: S^1 \to D^2$ denote the inclusion, then $\varphi i = 1$ and we have a commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\quad 1 \quad} & S^1 \\ & {}_{i}\searrow \quad \nearrow_{\varphi} & \\ & D^2 & \end{array}$$

This leads to another commutative diagram

$$\begin{array}{ccc} \pi(S^1,1) & \xrightarrow{\quad 1 \quad} & \pi(S^1,1) \\ & {}_{i_*}\searrow \quad \nearrow_{\varphi_*} & \\ & \pi(D^2,1) & \end{array}$$

But $\pi(D^2,1) = 0$, since D^2 is contractible, and so we get a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} \\ & {}_{i_*}\searrow \quad \nearrow_{\varphi_*} & \\ & 0 & \end{array}$$

which is impossible. This contradiction proves the result.

16.11 Exercises

(a) Given $[f] \in \pi(S^1,1)$, let γ be the contour $\{ f(t); t \in I \} \subset \mathbb{C}$ and define

$$w(f) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}$$

Prove that (i) $w(f)$ is an integer,
(ii) $w(f)$ is independent of the choice of $f \in [f]$,
(iii) $w(f) = \deg(f)$.

(b) Let $f\colon S^1 \to S^1$ be the mapping defined by $f(z) = z^k$ for some integer k. Describe $f_*\colon \pi(S^1,1) \to \pi(S^1,1)$ in terms of the isomorphism $\pi(S^1,1) \cong \mathbb{Z}$.

(c) Let α, β be the following closed paths in $S^1 \times S^1$.

$\alpha(t) = (\exp(2\pi i t), 1)$, $\qquad \beta(t) = (1, \exp(2\pi i t)$.)

Show, by means of diagrams, that $\alpha * \beta \sim \beta * \alpha$.

(d) Calculate $\pi(\underbrace{S^1 \times S^1 \times \dots \times S^1}_{n}, (1, 1, \dots, 1))$.

(e) Using Exercise 15.16(c) deduce that the torus is not homeomorphic to the sphere S^2.

(f) Prove that the set of points $z \in D^2$ for which $D^2 - \{ z \}$ is simply connected is precisely S^2. Hence prove that if $f\colon D^2 \to D^2$ is a homeomorphism then $f(S^1) = S^1$.

(g) Find the fundamental groups of the following spaces.
(i) $\mathbb{C}^* = \mathbb{C} - \{ 0 \}$;
(ii) $\mathbb{C}^*/G$, where G is the group of homeomorphisms $\{ \varphi^n; n \in \mathbb{Z} \}$ with $\varphi(z) = 2z$.
(iii) $\mathbb{C}^*/H$ where $H = \{ \psi^n; n \in \mathbb{Z} \}$ with $\psi(z) = 2\bar{z}$.
(iv) $\mathbb{C}^*/\{ e, a \}$, where e is the identity homeomorphism and $az = -\bar{z}$.