Dupin indicatrix at q_1 and q_2 are parallel, and their common direction r' is conjugate to r. We shall leave the proofs of these assertions to the Exercises (Exercise 12).

EXERCISES

1. Show that at a hyperbolic point, the principal directions bisect the asymptotic directions.

2. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

3. Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$|k| \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p.

4. Assume that a surface S has the property that $|k_1| \leq 1$, $|k_2| \leq 1$ everywhere. Is it true that the curvature k of a curve on S also satisfies $|k| \leq 1$?

5. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^{\pi} k_n(\theta)\, d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

7. Show that if the mean curvature is zero at a nonplanar point, then this point has two orthogonal asymptotic directions.

8. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:

a. Paraboloid of revolution $z = x^2 + y^2$.

b. Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.

c. Catenoid $x^2 + y^2 = \cosh^2 z$.

9. Prove that

a. The image $N \circ \alpha$ by the Gauss map $N\colon S \to S^2$ of a parametrized regular curve $\alpha\colon I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the sphere S^2 (called the *spherical image* of α).

b. If $C = \alpha(I)$ is a line of curvature, and k is its curvature at p, then

$$k = |k_n k_N|,$$

where k_n is the normal curvature at p along the tangent line of C and k_N is the curvature of the spherical image $N(C) \subset S^2$ at $N(p)$.

10. Assume that the osculating plane of a line of curvature $C \subset S$, which is nowhere tangent to an asymptotic direction, makes a constant angle with the tangent plane of S along C. Prove that C is a plane curve.

11. Let p be an elliptic point of a surface S, and let r and r' be conjugate directions at p. Let r vary in $T_p(S)$ and show that the minimum of the angle of r with r' is reached at a unique pair of directions in $T_p(S)$ that are symmetric with respect to the principal directions.

12. Let p be a hyperbolic point of a surface S, and let r be a direction in $T_p(S)$. Describe and justify a geometric construction to find the conjugate direction r' of r in terms of the Dupin indicatrix (cf. the construction at the end of Sec. 3-2).

***13.** (*Theorem of Beltrami-Enneper.*) Prove that the absolute value of the torsion τ at a point of an asymptotic curve, whose curvature is nowhere zero, is given by

$$|\tau| = \sqrt{-K},$$

where K is the Gaussian curvature of the surface at the given point.

***14.** If the surface S_1 intersects the surface S_2 along the regular curve C, then the curvature k of C at $p \in C$ is given by

$$k^2 \sin^2\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta,$$

where λ_1 and λ_2 are the normal curvatures at p, along the tangent line to C, of S_1 and S_2, respectively, and θ is the angle made up by the normal vectors of S_1 and S_2 at p.

15. (*Theorem of Joachimstahl.*) Suppose that S_1 and S_2 intersect along a regular curve C and make an angle $\theta(p)$, $p \in C$. Assume that C is a line of curvature of S_1. Prove that $\theta(p)$ is constant if and only if C is a line of curvature of S_2.

***16.** Show that the meridians of a torus are lines of curvature.

17. Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N\colon S \to S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$. Show that the above condition implies that the angle of two intersecting curves on S and the angle of their spherical images (cf. Exercise 9) are equal up to a sign.